A Simultaneous Prediction Limit on the Means of Future Samples from an Exponential Distribution

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This article provides a lower simultaneous prediction limit on the means of all of \(k\) future samples using the results of a previous sample from the same exponential distribution.

Such limits are required as specifications on future life for components, as warranty limits for the future performance of a specified number of systems with standby units, and in various other applications.

Exact and approximate limits are given and illustrated with a numerical example. The results are extended to situations where the past and/or future samples are censored. Some other prediction limits for the exponential distribution are also summarized. The mathematical justification of the results is briefly developed in an appendix.

The discussion is in the general context of life testing analysis.

1. **Problem Statement and Introduction**

Suppose that the times to failure for some product follow an exponential distribution. The given data consist of the failure times of \(m\) randomly selected units. It is desired to construct a lower simultaneous prediction limit that is exceeded with probability \(\gamma\) by the mean times to failure of all of \(k\) future samples, each consisting of \(n\) units.

Two situations where such limits are required are:

1. A customer has placed an order for a product which has an exponential time-to-failure distribution. The terms of his purchase call for \(k\) monthly shipments. From each shipment the customer will select a random sample of \(n\) units and accept the shipment only if the mean time to failure for this sample exceeds a specified lower limit. The manufacturer wishes to use the results of a previous sample of \(m\) units to calculate this limit so that the probability is \(\gamma\) that all \(k\) shipments will be accepted. It is assumed that the \(m\) past units and the \(nk\) future units are random samples from the same population. (For some examples of the use of the exponential distribution as a model in product life analysis, see Davis [3].)

2. A system consists of \(n\) identical components whose times to failure follow an exponential distribution. Initially one component is operating and the remaining \(n - 1\) components are in a standby mode; a new component goes into operation as soon as the preceding component has failed. The system is said to fail when all \(n\) components have failed. Thus the system time to failure is the total of the failure times for the \(n\) components and follows a gamma distribution (see Hahn and Shapiro [7]). A simultaneous lower prediction limit to be exceeded with probability \(\gamma\) by the mean times to failure of all of \(k\) future systems is desired. This limit is to be calculated from the times to failure of \(m\) previously tested components. (Similar problems also arise in various product maintenance and servicing problems.)

Section 2 provides the expressions for the desired simultaneous prediction limit. A numerical example is given in Section 3. Alternative conservative prediction limits are given in Section 4. The results are extended in Section 5 to the situation where the past and/or the future samples involve Type 2 censoring. Related previously obtained prediction limits for samples from an exponential distribution are briefly described in Section 6 and some concluding remarks are made in Section 7. The mathematical justification for the results is given in an appendix. The reader who desires a general survey...
of prediction intervals for various distributions is referred to Hahn and Nelson [8] and the references given there.

The discussion here is in the context of life testing to suggest the most frequent area of application. However, the results apply for any set of independently distributed exponential variates.

2. THE SIMULTANEOUS LOWER PREDICTION LIMIT

Let \( x_1, \ldots, x_m \) denote the observed times to failure of \( m \) randomly selected units from an exponential distribution. Also let

\[ \bar{x} = \frac{\sum_{i=1}^{m} x_i}{m} \]

denote the mean time to failure for these units. Let \( k \) further samples each of size \( n \) be selected from the given population, independently of one another and of the first sample. Then a lower 100\( \gamma \)% simultaneous prediction limit to be exceeded by the mean times to failure for all of the \( k \) future samples is

\[ \bar{x} w(k, 2n, 2m; 1 - \gamma), \]

where \( w(k, v_1, v_0; 1 - \gamma) \) is the 100(1 - \( \gamma \)) percentage point of the studentized smallest chi-square distribution for \( k \) variates each with \( v_1 \) degrees of freedom and one variate with \( v_0 \) degrees of freedom, i.e., the distribution of

\[ v_0 \min (z_1, \ldots, z_k)/(v_0 z_0), \]

where \( z_0, z_1, \ldots, z_k \) are all independent chi-square variates with \( v_0, v_1, \ldots, v_k \) degrees of freedom, respectively. The factors \( w(k, v_1, v_0; 1 - \gamma) \) are tabulated by Krishnaiah and Armitage [14] for all combinations of \( k = 1(1)12, v_1 = 1(1)20, v_0 = 5(1)45 \) and \( 1 - \gamma = 0.01, 0.025, 0.05 \) and \( 0.10 \).

In the case where the \( k \) future samples are based on varying numbers of observations, \( n_1, \ldots, n_k \), an approximate conservative lower simultaneous prediction limit is obtained by using \( \min(n_i) \) in place of \( n \) in the expression given above. This limit is conservative in the sense that it underestimates the true lower 100\( \gamma \)% simultaneous prediction limit. Also exact results can be obtained from the multivariate \( F \) distribution (see Appendix) using numerical procedures described by Amos and Bulgren [1].

The preceding simultaneous prediction limit has the following interpretation. Suppose that for many sets of a single past sample and \( k \) future samples, such lower 100\( \gamma \)% simultaneous prediction limits are calculated. Then, in the long run, in a fraction \( \gamma \) of the many such sets of \( k + 1 \) samples, all of the \( k \) future means will, in fact, exceed the lower prediction limit calculated from the results of the first sample.

3. A NUMERICAL EXAMPLE

Assume that the observed mean time to failure in a life test on \( m = 10 \) units from an exponential distribution is \( \bar{x} = 4200 \) hours. Three further samples \( (k = 3) \), each of \( n = 5 \) units, are to be selected randomly from the same population. A lower 95\% simultaneous prediction limit on the mean times to failure for all of the three future samples is

\[ 4200(0.2641) = 1109 \text{ hours}, \]

since \( w(3, 10, 20; 0.05) = 0.2641 \) from Krishnaiah and Armitage [14].

4. CONSERVATIVE LOWER SIMULTANEOUS PREDICTION LIMITS

Two conservative procedures for obtaining a lower 100\( \gamma \)% simultaneous prediction limit on the mean times to failure for all of \( k \) future samples each of size \( n \), based on a past sample of size \( m \) from the same exponential distribution with observed mean failure time \( \bar{x} \), are

Approximation 1: \( \bar{x}/F(2m, 2n; \gamma^{1/2}) \)

and

Approximation 2: \( \bar{x}/F[2m, 2n; 1 - ((1 - \gamma)/k)] \),

where \( F(v_1, v_0; \gamma) \) denotes the 100\( \gamma \)'th percentage point of the \( F \)-distribution with \( v_1 \) degrees of freedom in the numerator and \( v_0 \) degrees of freedom in the denominator. Approximation 1 usually requires percentage points of the \( F \)-distribution not given in the standard tabulations and Approximation 2 often does. To obtain such values one can use the well-known relationship between the \( F \) and beta distributions (see Mood, Graybill and Boes [20], p. 249) and the tabulations of the incomplete beta function (Pearson [22]) or, for improved precision, a direct computer evaluation of the incomplete beta function. The preceding expressions are useful for situations outside the Krishnaiah and Armitage tabulations or if these tabulations are not readily available.

Approximation 1 is more accurate than Approximation 2; that is, it is closer to the exact answer and therefore less approximate. On the other hand, Approximation 2 may sometimes be more convenient to apply in situations, such as when \( \gamma = 0.95 \) and \( k = 2 \), where the standard \( F \) tables can be used for Approximation 2, but not for Approximation 1.

For the numerical example in Section 1, the conservative lower 95\% simultaneous prediction limit to be exceeded by each of the three future samples of size 5 is found from Approximation 1 to be 4200/3.816 = 1101 hours, since \( F(20, 10; (0.95)^{1/2}) = F(20, 10; 0.98305) = 3.816 \) and from Approximation 2 is found to be 4200/3.835 = 1095 hours, since \( F[20, 10; 1 - ((1 - 0.95)/3)] = F(20, 10; 0.98333) \)

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These values are close to the value of 1109 hours obtained from the exact method.

More generally, Table 1 shows the percent by which the two approximate procedures underestimate the exact lower 95% simultaneous prediction limit. This tabulation indicates that the approximate procedures provide reasonably good results unless \( k \) and \( n \) are large and \( m \) is small. The approximate procedures are somewhat better for constructing 99 percent prediction limits and somewhat poorer for 90 percent prediction limits than for 95 percent prediction limits.

In the case where the \( k \) future samples are based on a varying number of observations, \( n_1, \cdots, n_k \), a further conservative approximation is to use \( \min, n \), in place of \( n \) in the expressions given above.

5. EXTENSION OF RESULTS TO CENSORED SAMPLES

The preceding results can be readily extended to the situation where the past sample and/or the future samples are Type 2 censored. In particular, assume that the past life test involves the simultaneous placement of \( m \) units on test, but that the life test is terminated when a predetermined number, \( m' \leq m \), of failures has taken place. Let \( x_1, \cdots, x_m \) denote the \( m' \) observed failure times; in addition, there are \( m - m' \) units each with running time \( x_m \) at the time the life test is terminated. As previously, it is assumed that time to failure is exponentially distributed. Now let \( k \) future samples, each of size \( n \), be selected from the same population independently of one another and of the first sample and placed on life test until a predetermined number, \( n' \leq n \), failures have occurred on each of these samples. The mean time to failure for a future sample will then be estimated, using standard procedures (see Epstein and Sobel [5]), to be

\[
\bar{y} = \left[ \sum_{i=1}^{n'} x_i + (n - n')y_m \right] / n',
\]

where \( y_1, \cdots, y_m \) are the times to failure for the \( n' \) failed units of the future sample.

Then a lower 100\( \gamma \% \) simultaneous prediction limit to be exceeded by the estimated mean times to failure for all of the \( k \) future samples is

\[
\bar{x}'w(k, 2n', 2m'; 1 - \gamma),
\]

where

\[
x' = \left[ \sum_{i=1}^{n} x_i + (m - m')x_m \right] / m'
\]

is the mean time to failure of the past sample.

Alternately, conservative lower 100\( \gamma \% \) simultaneous prediction limits for the preceding situation are

Approximation 1: \( \bar{x}/F(2m', 2n'; \gamma^{1/k}) \)

and

Approximation 2: \( \bar{x}/F(2m', 2n'; 1 - ((1 - \gamma)/k)) \).

6. SOME RELATED RESULTS

Prediction intervals for future samples from various distributions have been developed in recent years. A survey of these results is provided by Hahn and Nelson [8]. For the exponential distribution, Nelson [21] has shown that a lower 100\( \gamma \% \) simultaneous prediction limit to be exceeded by the smallest observation in \( k \) future samples, each of size one, given a sample mean \( \bar{x} \) based on \( m \) previous observations from the same population, is

\[
\bar{x}/[kF(2m, 2; \gamma)].
\]

This result is a special case, with \( n = 1 \), of the lower 100\( \gamma \% \) simultaneous prediction limit given here. Lawless [16] extended the preceding results to obtain a lower simultaneous prediction limit that is to be below the values of at least \( k' < k \) of the future samples of size one and discusses the application of these results to system reliability problems.

Nelson [21] also essentially shows that a lower

Table 1—Percent by which the two approximate procedures underestimate the exact lower 95% simultaneous prediction limit on the mean times to failure of all \( k \) future samples of size \( n \), given a past sample of size \( m \)

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The two values shown are the percent underestimates using Approximations 1 and 2, respectively.
100γ% simultaneous prediction limit to be exceeded by the means of the minimum values of each of k future samples, each of size n, from an exponential distribution, given a sample mean \( \bar{x} \) based on m previous observations from the same population, is

\[
\frac{\bar{x}}{\sqrt{nF(2m, 2k; \gamma)}}.
\]

The preceding result differs from the one discussed here in that it provides a lower simultaneous prediction limit on the mean of the minima of k future samples each of size n, while the present results provide lower simultaneous prediction limits on the minima of the means of k future samples each of size n. Both problems are of interest in practice.

Additional results dealing with prediction intervals for the exponential distribution are given in papers by Dykstra, Hewett and Thompson [4], Faulkenberry [6], Hall and Prairie [9], Hewett [10], Hewett and Bulgren [11], Kaminsky and Nelson [12] and [13], Lawless [15] and Likes [17] and in the book by Mann, Schafer and Singpurwalla [18]; Hahn and Nelson [8] give further details on most of these papers.

7. Concluding Remarks

Simultaneous prediction limits are required in many practical situations. This article has provided a lower simultaneous prediction limit for the future means of samples from an exponential distribution. These methods can be readily extended to obtain similar limits for a sample from a Weibull distribution with a known value of the shape parameter. This is done with a power transformation of the failure times, as described by Hewett and Bulgren [11] and by Lawless [16].

Also we have considered only lower simultaneous prediction limits for an exponential distribution since those are what are required in practice most frequently. However, upper simultaneous prediction limits can be developed in a similar manner. These would require use of the percentage points of studentized largest chi-square variates. Such factors have been tabulated by Armitage and Krishnaiah [2]. An upper prediction limit might be of interest, for example, if one desires an upper bound on the total time it will take k future customers to pass through a service shop involving n identical independent operations which take place immediately after one another.

8. Appendix: Justification of Results

The mathematical justification of the results of Section 5 are given here. The results of Sections 2 and 4 are special cases of those of Section 5.

Let \( X_1, \ldots, X_m \) denote the first \( m' \) ordered observations from a past random sample of size \( m \) from an exponential distribution with a mean \( \theta \), i.e., a distribution with probability density

\[
f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x \geq 0.
\]

Furthermore, let \( Y_{i1}, \ldots, Y_{in} \), \( i = 1, \ldots, k \), denote the first \( n' \) ordered observations from each of \( k \) future random samples of size \( n \) from the same population, where the \( k \) samples are selected independently of one another and of the first sample.

Now let

\[
\bar{X}' = \left[ \sum_{i=1}^{m'} X_i + (m - m')X_m \right] / m'
\]

and

\[
\bar{Y}_i' = \left[ \sum_{j=1}^{n'} Y_{ij} + (n - n')Y_{in} \right] / n',
\]

\( i = 1, \ldots, k \)

denote the usual estimates of the mean \( \theta \) from each of the \( (k + 1) \) censored samples (see Epstein and Sobel [5]). It follows (see [5]) that \( 2m'\bar{X}'/\theta \) is a chi-square variate with \( 2m' \) degrees of freedom and that \( 2n'\bar{Y}_i'/\theta \), \( i = 1, \ldots, k \), are each chi-square variates with \( 2n' \) degrees of freedom. Also each of the \( (k + 1) \) chi-square variates are independent of one another. Thus, \( \min_i (\bar{Y}_i')/\bar{X}' \) is distributed as \( W(k, 2n', 2m') \) where \( W(k, 2n', 2m') \) is a studentized smallest chi-square variate for \( k \) (numerator) chi-square variates with \( v_1 \) degrees of freedom each and one (denominator) chi-square variate with \( v_0 \) degrees of freedom.

Therefore one can make the probability statement

\[
Pr \left[ \min_i (\bar{Y}_i')/\bar{X}' \geq W(k, 2n', 2m'; 1 - \gamma) \right] = \gamma,
\]

where \( W(k, v_1, v_0; 1 - \gamma) \) is the \( 100(1 - \gamma) \) percentage point of the distribution of \( W(k, v_1, v_0) \).

Equivalently,

\[
Pr \left[ \min_i (\bar{Y}_i') \geq \bar{X}'w(k, 2n', 2m'; 1 - \gamma) \right] = \gamma
\]

and this provides the desired lower 100γ% simultaneous prediction limit to be exceeded by each of the \( k \) future sample means.

The conservative lower 100γ% simultaneous prediction limits given in Section 5 follow from the fact that \( \bar{Y}_i'/\bar{X}' \), \( i = 1, \ldots, k \), are each distributed as \( F \) variates with \( 2n' \) and \( 2m' \) degrees of freedom in the numerator and denominator, respectively, and from results obtained by Dykstra, Hewett and Thompson [4] (Approximation 1) and the Bonferroni Inequality, see Miller [19], (Approximation 2).

The fact that the \( \bar{Y}_i'/\bar{X}' \) follow a multivariate \( F \) distribution is also the basis for obtaining simultaneous prediction limits when the number of future observations differ, using the evaluation procedures described by Amos and Bulgren [1].

9. Acknowledgment

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