Theory for Optimum Censored Accelerated Life Tests for Normal and Lognormal Life Distributions

Wayne Nelson and Thomas J. Kielpinski

General Electric Company
Corporate Research and Development
Schenectady, New York 12345

6514 Test Squadron
Hill Air Force Base, Utah 84406

This expository paper presents theory for optimum plans for accelerated life tests for estimating a simple linear relationship between a stress and product life, which has a normal or lognormal distribution, when the data are to be analyzed before all test units fail. Standard plans with equal numbers of test units at equally spaced test stresses are presented and are compared with the optimum plans. While the optimum plans may not always be robust enough in practice, they indicate that more test units should be run at low stress than at high stress. The plans are illustrated with a temperature-accelerated life test of an electrical insulation analyzed with the Arrhenius model.

KEY WORDS
Optimum Plans
Accelerated Testing
Censored Life Data
Overstress Testing
Normal and Lognormal Distributions
Reliability
Maximum Likelihood Estimation

1. INTRODUCTION

Accelerated tests quickly provide information on the life of products and materials. Test units are subjected to severe stresses that yield shorter lives. Accelerated conditions are produced by high temperature, voltage, pressure, vibration, cycling rate, load, etc. The life data from the high stresses are extrapolated to estimate the life distribution at normal design conditions.

Such tests save time and expense over tests at normal conditions. Further savings result from analyzing the test data before all units fail. Then the data consist of times to failure on failed units and running times on unfailed units. Such censored data may be analyzed as described later. Thus one can analyze the most recent test data while test units are still running. Also, one may terminate the test at a specified time and analyze the data.

Further savings result from a good test plan. A plan specifies the test stresses and the number of test units at each stress. Plans commonly use equally spaced test stresses, each with the same number of test units. Such standard plans are usually inefficient for estimating product life at a low design stress. An optimum plan provides more accurate estimates for the same number of test units and test time.

Literature. There is much literature on optimum test plans for least squares fitting of linear regression models to complete data (i.e., all units are run to failure). For the simple linear regression model considered here, Gaylor and Sweeney [5] summarize the literature. Herzberg and Cox [11] briefly survey key papers on optimum design for multiple linear regression models, and Chernoff [3] generally surveys optimal design in experiments. The limited literature on optimum plans for fitting regression models to censored data is briefly summarized below. It deals with a variety of models (distributions and relationships), methods of estimation, types of censoring, and test constraints.

Two constraints have been used. For one, all test units are started and run simultaneously. When the units reach a specified running time, the resulting censored data are analyzed. For the other, the test units are run successively one after the other, and the data are analyzed when a specified total running time is accumulated. Such testing is necessary, if the test equipment runs only one test unit at a time.

Two types of censoring have been considered. Type I censoring involves running each unit a prespecified time unless it fails sooner. Then the censoring times are fixed, and the number of failures is random. Type II censoring involves simultaneous
testing of the units until a prescribed number of them fail. Then the common censoring time is random, and the number of failures is fixed.

Chernoff [2] considers maximum likelihood estimation of the failure rate of an exponential distribution at a design stress. His relationships for the failure rate are 1) a quadratic function of a stress and 2) an exponential function of a stress. He gives optimum plans both for simultaneous testing with Type I censored data and for successive testing with complete data.

Little and Jebe [13] consider least squares estimation of the mean of a normal distribution at a design stress. The mean is a simple linear regression function of a stress and the standard deviation is a constant. They give optimum plans for successive testing with complete data.

Mann [14] considers linear estimation with order statistics to estimate a percentile of an extreme value (or Weibull) distribution at a design stress. The percentile is a multiple linear regression function of a number of stresses, and the scale parameter is a constant. She gives approximately optimum plans for Type II censored data.

Meeker and Nelson [15], [16] consider maximum likelihood estimation of a percentile of an extreme value (or Weibull) distribution at a design stress. The percentile is a simple linear regression function of a stress, and the scale parameter is a constant. They give optimum plans for simultaneous testing with Type I censored data.

Purpose. This paper considers maximum likelihood estimation of the mean of a normal (or lognormal) distribution at a design stress. The mean is a simple linear regression function of a stress, and the (log) standard deviation is a constant. This paper provides optimum plans for simultaneous testing with Type I censored data. Nelson and Kielpinski [20] provide more details.

Contents. The following are the contents of this paper. Section 2 presents the accelerated testing model. Section 3 describes the problem and the maximum likelihood method of estimation. Section 4 presents the best “standard” plans with equally spaced test stresses, each with the same number of test units. Section 5 presents the optimum plans. Section 6 compares the standard and optimum plans and suggests compromise plans. Section 7 provides theory for the plans and a Monte Carlo verification of the theory.

2. The Model

This section describes the assumed model for product life as a function of a stress. This model includes the Arrhenius model, which is used in the example.

The General Model

The assumptions of the general model are:

i) At any stress, product life (or the logarithm of life) has a normal distribution.

ii) The distribution standard deviation $\sigma$ has the same value at any stress.

iii) The distribution mean $\mu$ is a linear function of a (possibly transformed) stress $x$; that is,

$$\mu(x) = \alpha + \beta x. \quad (2.1)$$

The model parameters $\alpha$, $\beta$, and $\sigma$ are to be estimated from test data. Also, it is assumed that the random variations in the lives of the test units are statistically independent. This is a standard simple linear regression model. The mean (2.1) is also the 50th percentile and is commonly used as a typical life. The plans presented below provide the most precise estimate of the mean at a specified design stress.

The Arrhenius Model

The Arrhenius model is often used for temperature-accelerated life tests of materials that fail from chemical degradation. The assumptions of the model are:

i) At any temperature, product life has a log-normal distribution (the logarithm of life has a normal distribution).

ii) The standard deviation $\sigma$ of the logarithm of life is a constant (i.e., independent of temperature).

iii) The mean $\mu(x)$ of the logarithm of life is a linear function of the reciprocal $x = 1000/T$ of the absolute temperature $T$; that is,

$$\mu(x) = \alpha + \beta x \quad (2.2)$$

where $\alpha$ and $\beta$ are parameters characteristic of the product and the test method.

The antilogarithm of the so-called logarithmic mean (2.2) is the median life. Equation (2.2) is loosely called the Arrhenius relationship. This model is presented in more detail by Nelson [17].

This model is depicted in Figure 1 on Arrhenius paper. Such paper has a transformed temperature scale and a log scale for time. A linear scale for reciprocal absolute temperature was added to the figure. The Arrhenius relationship (2.2) is a straight line on such paper.

If temperature is regarded as the stress variable, then product life decreases with increasing stress. However, if reciprocal absolute temperature is regarded as the variable in the relationship, then life decreases with decreasing “reciprocal” stress. Hereafter, “high stress” simply means test conditions that produce short life.
An Example

The following example illustrates the test plans presented later.

To evaluate a new Class-B insulation for electric motors, a temperature-accelerated life test was run. The purpose was to estimate the median life of such insulation at the design temperature of 130°C. Ten motorettes with the insulation were run at each of four test temperatures (150°C, 170°C, 190°C, and 220°C). The data were to be analyzed when the units had run for 8064 hours. The data are censored at 8064 hours and are given by Crawford [4] and are plotted in Figure 1. The use of such data plots is described in detail by Hahn and Nelson [8].

Crawford [4] fitted the Arrhenius model to the data by maximum likelihood and obtained the relationship between life and temperature shown in Figure 1. His estimates of the model parameters are needed later and are \( \alpha = -6.014 \), \( \beta = 4.3070 \) and \( \delta = 0.2590 \), which were obtained from the data with the computer program of Hahn and Miller [7]. The analysis used common (base 10) logarithms of life.

3. The Problem

This section fully describes the constraints, the maximum likelihood method for censored data, and the optimization criterion.

A test plan for the model above specifies the test stresses, the number of test units, and the proportion of units at each test stress.

Constraints

The test plans are subject to the following constraints.

It is assumed that all test units start at the same time and run simultaneously without replacement, a common practice. It is also assumed that the test is truncated. That is, the data are to be analyzed when the units have been on test a prechosen time \( t \). This truncation or censoring time is determined by schedules and costs. This time should be long to maximize the information from the test. A test might be continued beyond \( t \) to provide further data for a later analysis.

The highest allowable test stress must be specified. If this stress were not specified, its optimum value would be infinitely large. In practice, the higher this stress, the more informative the test plan, provided that the model is satisfactory over the range of stress. Thus the highest stress should not cause failure modes different from those at the design stress.

Estimation Methods

Various methods for censored data provide estimates of the parameters of the assumed regression model. However, ordinary least squares estimation does not apply to censored data. Applicable methods include maximum likelihood estimation (ML), linear estimation based on order statistics, and graphical estimation. These methods are compared by Hahn and Nelson [9].

Maximum likelihood estimation is used here for the following reasons. 1) The optimum plans are easier to calculate than those for linear estimation. 2) Available computer programs do the laborious ML calculations for such censored data (Glasser [6], Hahn and Miller [7], and Nelson and Hendrickson [19]). There are no such programs for the laborious calculations for linear estimation. 3) ML estimation provides asymptotically minimum variance estimates. That is, the large-sample variance of the ML estimate of the mean at a specified stress is no greater than the variance of any other estimate. Also, for small sample sizes, ML estimates generally compare well with other estimates. 4) Linear estimation is designed for data with Type II censoring. Thus it is not strictly correct for the Type I censoring considered here, whereas ML estimation is. 5) The optimum design for ML estimation can be expected to be close to optimum for other methods of estimation, even the graphical method.
The Optimization Criterion

An optimum test plan provides a “best” estimate of the model. Criteria for “best” are reviewed by Nelson [17]. The criterion here is that the large-sample variance of the ML estimate of the mean at a specified (design) stress be minimized. Plans which minimize this variance are presented below. If the life distribution is lognormal, this criterion minimizes the relative (percentage) error in the estimate of the median life.

For example, the insulation test was to estimate the median life at the design temperature of 130°C. For practical reasons, the highest test temperature was chosen to be 220°C. A data analysis was required after 8064 hours. Test units started test at different times and thus have different censoring times. However, they could have been started together to yield more information earlier. For illustrative purposes, it is assumed that all units started test together. Following sections present test plans that meet these constraints and are more informative than the actual one.

4. Best Standard Test Plans

The Problem

This section presents best standard test plans. Standard plans are commonly used and have K equally spaced test stresses each with the same number of test units. They are presented for later comparison with the optimum plans.

It is assumed that the highest test stress \( x_H \) has been specified. The lowest test stress \( x_L \) must be specified by the plan. The remaining test stresses are equally spaced between the highest and lowest stresses. A fraction \( 1/K \) of the sample units is tested at each stress. The low test stress of the best plan minimizes the large-sample variance of the ML estimate of the mean at the specified design stress \( x_s \).

The Best Test Stress

The best low test stress for a standard plan is

\[
x_L = x_H + \xi_K (x_s - x_H). \tag{4.1}
\]

Here \( \xi_K \) is a function of \( K \) and the model parameters; it is obtained as shown in Section 7. Also, Nelson and Kielpinski [20] give simple charts for \( \xi_K \). A transformed value \( x_L \) must be converted to the accelerating stress value.

For the insulation example, the reciprocal value of the highest test temperature of 220°C is \( x_H = 1000/(220 + 273.16) = 2.0277 \), and the reciprocal value of the design temperature of 130°C is \( x_s = 2.4804 \). For \( K = 4 \) test stresses, the best value is \( \xi_4 = 0.72 \). Consequently, the best lowest stress has the reciprocal value \( x_L = 2.0277 + 0.72(2.4804 - 2.0277) = 2.3536 \). In degrees Centigrade, this is \( T = (1000/2.3536) - 273.16 = 152°C \). The actual lowest test temperature was 150°C. Also, the four test temperatures were not quite equally spaced and their censoring times were not the same. However, this may be ignored for the purpose of an example.

The lowest test temperature of 152°C is best only for the censoring time of 8064 hours. However, the test was not terminated then when the corresponding data were analyzed. It was terminated at 17,661 hours when just three units at 150°C were still running. For this censoring time, the best lowest test temperature would be 140°C. In such situations with a number of censoring times, one would choose a compromise censoring time and corresponding best plan.

The theory for best standard plans uses equally spaced values of transformed stress. That is, for the Arrhenius model, equally spaced reciprocal absolute temperatures are used. Equally spaced temperatures are more common in practice. For practical purposes, the difference can be ignored in the example.

For any number of test stresses, the maximum value of \( \xi_K \) is 2. Then the test stresses are located symmetrically above and below the design stress. This occurs for large censoring times and is the optimum plan for complete data (Gaylor and Sweeny [5]). Such testing below the design stress requires long test times and is impractical.

Variance of the Estimate

For a best plan with a sample size \( n \), the large-sample variance of the ML estimate \( \hat{\mu}_s \) of the design mean is

\[
\text{Var} (\hat{\mu}_s) = \sigma^2 V_{x_L}/n \tag{4.2}
\]

where the variance factor \( V_{x_L} \) is a function of the number \( K \) of test stresses and the model parameters. For example, for the insulation test with a standard plan with four test stresses, \( V_4 = 10.2 \) and is obtained as described in Section 7. Eq. (4.2) may be used to determine an appropriate sample size \( n \).

Nelson and Kielpinski [20] provide charts for the variance factors for \( K = 2, 3, \) and 4 test stresses. The minimum possible value of the variance factor is 1. This minimum corresponds to the most informative situation where each observation is uncensored and at the design stress.

For practical situations, the best standard plan with two stresses is more precise than that with three stresses, which in turn is more precise that with four stresses (Nelson and Kielpinski [20]).

5. Optimum Test Plans

The Problem

This section describes optimum test plans that
use just two test stresses. The optimum plan minimizes the large-sample variance of the ML estimate of the mean at a specified design stress $x_0$. It is assumed that the high test stress $x_H$ has been specified. It should be as high as practically possible in order to minimize the variance. The low test stress and the proportion of the test units allocated to it are chosen to optimize the plan. The remaining test units are tested at the high stress. The following gives the methods for finding the optimum low test stress and the optimum allocation.

The optimum plan has good properties for other purposes, for example, for comparing a number of products at a common design stress. Each product would be tested with an optimum plan to obtain the most precise estimate of the mean at the common design stress.

**The Optimum Test Stresses**

The optimum low test stress is

$$x_L = x_H + \xi(x_0 - x_H).$$

Here $\xi$ is a function of the censoring time and the model parameters; it is obtained as shown in Section 7. Also, Nelson and Kielpinski [20] give a chart for $\xi$. A transformed value $x_2$ must be converted to the accelerating stress value.

For the insulation test, the method in Section 7 yields the value $\xi = 0.63$. Then the optimum stress is $x_L = 2.0277 + 0.63(2.4804 - 2.0277) = 2.3129$. In Centigrade degrees, this is $T = (1000/2.3129) - 273.16 = 159^\circ C$. The actual lowest test temperature was $150^\circ C$.

The lowest test temperature of $159^\circ C$ is optimum only for the censoring time of 8064 hours when the available data were analyzed. However, the test was terminated at 17,661 hours, when just three units at $150^\circ C$ were still running. For this censoring time, the optimum low test temperature is $140^\circ C$.

In such situations with a number of censoring times, one would choose a compromise censoring time and its optimum plan.

**The Optimum Allocation**

The optimum proportion $p^*$ of test units to run at the optimum low test stress is obtained with the method described in Section 7. Also, Nelson and Kielpinski [20] provide a chart for $p^*$. The chart shows that in practical applications, more units should be run at the low test stress.

For the insulation test, the method in Section 7 yields the value $p^* = 0.735$. That is 73.5% of the units would be tested at $150^\circ C$. For a sample size of 40, $(0.735)40 \equiv 29$ units would be tested at $150^\circ C$, and the other 11 units would be tested at $220^\circ C$.

**Variance of the Estimate**

For an optimum plan with a sample size $n$, the large-sample variance of the estimate $\hat{\mu}_0$ of the design mean is

$$\text{Var} (\hat{\mu}_0) = \sigma^2 V^*/n$$

where the variance factor $V^*$ is a function of the censoring time and model parameters. For the example, $V^* = 6.5$ and is obtained as described in Section 7. Nelson and Kielpinski [20] give a chart for the variance factor $V^*$. Eq. (5.2) may be used to determine an appropriate sample size.

6. **Comparison of the Optimum and Best Standard Plans**

This section compares the optimum and best standard plans with respect to 1) their precision (variances (4.2) and (5.2)), and 2) their robustness to an inadequate model or data. Compromise plans for both good precision and robustness are suggested.

**Comparison of Precision**

The following compares the plans with respect to the large-sample variance of the ML estimate of the design mean.

An example is provided by the insulation test. Suppose the optimum plan and the best standard plan with four test stresses are to be compared. For censoring at 8064 hours, the ratio of the variances of the estimates of the two plans given above is $10.2/6.5 = 1.57$. Thus the best standard plan with four stress requires 57% more test units than the optimum plan to achieve the same precision.

Nelson and Kielpinski [20] present charts that compare the large-sample variances of the optimum and best standard plans. The charts, of course, show that the optimum plan is always more precise than the best standard plans.

**Comparisons of Test Stresses and Allocations**

The charts of Nelson and Kielpinski [20] show the optimum plan and the standard plan with two test stresses have an interesting similarity. For most practical situations, the two plans have essentially the same low test stress. However, the optimum plan allocates more units to the low test stress. Also, the optimum plan requires a lower low test stress than does the standard plan.

**Robust Plans**

Optimum and best standard plans are not always suitable. Compromise plans that are better suited to applications are indicated below.

The optimum plan, of course, provides the smallest variance for the maximum likelihood estimate of the design mean. However, the optimum plan is
suitable only if the model is correct and the data are all good. However, a plan should be robust; that is, it should give useful results when the model is inaccurate or when some data are bad. Also, the plan should provide checks on the adequacy of the model and the data; methods for checks are presented by Nelson [17], Nelson and Hendrickson [19], and Hahn and Nelson [18]. Such checks require enough failures at three or more test stresses. The optimum plan is not robust. Standard plans with three or more test stresses tend to be robust, but they are less precise than the optimum plan.

To use an optimum plan, one must provide approximate values of the model parameters. Chernoff [1], [2], [3] calls such a plan "locally optimum." Values that are appreciably in error may result in a plan far from optimum. This possibility can be checked if one examines the plans for different parameter values. Also, one can vary the censoring time to examine the effect of analyzing the data before the planned censoring time and of running beyond it.

Robust plans have been suggested for regression analysis of complete data. Stigler [21] surveys some of the literature. Many compromise plans are based on assumptions that one may prefer to avoid in practice, for example, that all data are valid. Also, such plans usually compromise between good precision and robustness to one difficulty but neglect other important difficulties. Thus, good compromise plans must at present be determined by subjective judgment aided by the results on the optimum and best standard plans.

The model parameters can be estimated only if there are at least two distinct failures at one test stress and at least one failure at another test stress. If the low test stress of an optimum plan is too low, there may not be enough failures by the censoring time. This problem may be avoided through the use of a third test stress that is between the two optimum stresses and that is sure to yield failures.

**Compromise Plans**

The optimum plan suggests that a plan should use two test stresses and more units at the lower stress. A robust plan should use at least three test stresses that are sure to yield failures by the censoring time. Thus a robust compromise plan should use as few test stresses as possible but not less than three. For precision, the lowest test stress should have the greatest number of units, and the middle test stress should have the least number.

The following is suggested. The lowest test stress for a compromise plan should be between the lowest test stress for the optimum plan and that for the best standard plan with three or four test stresses. Also, the relative numbers of units for the two stresses for the optimum plan could be used at the extreme stresses for a compromise plan. The proportions of the sample for intermediate stresses should be small.

For such a compromise plan, the variance of the estimate of the design mean would generally be between the variances for the optimum and best standard plans. The variance of a compromise plan may be obtained from the theory presented in Section 7.

In conclusion, a compromise plan is generally preferable to the optimum plan, unless the model, the preliminary values of the parameters, and the data will be satisfactory.

7. **Theory**

This section provides asymptotic (large sample) theory for the optimum plans for a normal distribution (the theory applies to the lognormal distribution in the obvious way.) It presents maximum likelihood theory for estimation of the model and the expression for the large-sample variance of the estimate of the mean at a design stress. The variance is expressed in terms of the test stresses and the allocation of the test units to the stresses. The variances can then be optimized.

This presentation of the ML theory begins with a description of a conveniently reparametrized model. The presentation then provides the sample likelihood, the likelihood equations, the Fisher information matrix, and the covariance matrix of the ML estimates for a general plan. These results are then applied to standard and optimum plans to obtain the large-sample variance of the estimate of the mean at a design stress. This variance is minimized by the proper choice of the test stresses and the allocation. Finally, the theory is verified with Monte Carlo simulation.


**Reparametrized Model**

The following reparametrized model is convenient. Define the transformed stress

$$
\xi_i = \frac{x_i - x_n}{x_n - x_h}.
$$

(7.1)

As before $x_h$ is the value of the highest test stress, which is specified, and $x_n$ is the value of the design stress where the mean is to be estimated. For $x = x_h$, $\xi_i = 0$; and, for $x = x_n$, $\xi_i = 1$. The relationship (2.1) for the mean may be written in terms of $\xi_i$ as

$$
\mu(\xi_i) = \beta_n + \beta_i \xi_i.
$$

(7.2)
where new coefficients $\beta_0$ and $\beta_1$ are related to the previous $\alpha$ and $\beta$ by

$$\beta_0 = \alpha + \beta x_H - \mu_H \quad (7.3)$$

and

$$\beta_1 = \beta (x_S - x_H) - \mu_S - \mu_H \quad (7.4)$$

Here $\mu_S$ and $\mu_H$ are the means for $x_S$ and $x_H$, respectively. The mean $\mu_S = \beta_0 + \beta_1$ is to be estimated. The standard deviation $\sigma$ is the same for both forms of the model. To simplify the derivation, (7.2) is written as

$$\mu(\xi_i) = \beta_0 \xi_i + \beta_1 \xi_i \quad (7.5)$$

where $\xi_0 = 1$.

Log Likelihood

The log likelihood of an observation $y$ (time to failure) at a (transformed) stress $\xi_i$ and with Type I censoring follows. First define the indicator function $I = I(y)$ in terms of the censoring time $\eta$ by

$$I = \begin{cases} 1 & \text{if } y \leq \eta, \text{ failure observed by time } \eta, \\ 0 & \text{if } y > \eta, \text{ censored at time } \eta. \end{cases} \quad (7.6)$$

Let $z_1 = (y - \mu(\xi_i))/\sigma = (y - \beta_0 \xi_i - \beta_1 \xi_i)/\sigma$ be a standardized failure time, and let $z_2 = (\eta - \mu(\xi_i))/\sigma = (\eta - \beta_0 \xi_i - \beta_1 \xi_i)/\sigma$ be a standardized censoring time. Also let $\varphi = \varphi(\xi) = \varphi((\eta - \beta_0 \xi_i - \beta_1 \xi_i)/\sigma)$ and $\Phi = \Phi(\xi) = \Phi((\eta - \beta_0 \xi_i - \beta_1 \xi_i)/\sigma)$ where $\varphi(\cdot)$ is the density and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

The log likelihood $L$ of a Type I censored observation at a stress $\xi_i$ is

$$L = I[- \ln (\sigma) - (1/2) \ln (2\pi) - (1/2) z_1^2]$$

$$+ (1 - I) \ln (1 - \Phi). \quad (7.7)$$

Suppose the $i$th observation $y_i$ corresponds to a value $\xi_i$, and the corresponding log likelihood is $L_i$. Then the sample log likelihood $L_n$ for $n$ independent observations is

$$L_n = L_1 + \cdots + L_n \quad (7.8)$$

This is a function of the $y_i$, $\eta$, and $\xi_i$, and of the parameters $\beta_0$, $\beta_1$, and $\sigma$. The ML estimates $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\sigma}$ are the parameter values that maximize the sample log likelihood (7.8). Programs that calculate the estimates are referenced above.

Fisher Information Matrix

The second partial derivatives of the sample log likelihood with respect to the model parameters are needed. For a single observation, the three first partial derivatives are

$$\frac{\partial L}{\partial \beta_j} = \frac{\xi_i}{\sigma} \left\{ (\xi_i + (1 - I) \varphi - \varphi) \right\}, \quad j = 0, 1, \quad (7.9)$$

$$\frac{\partial L}{\partial \sigma} = \frac{1}{\sigma} \left\{ (\xi_i^2 - 1) + (1 - I) \frac{\xi_i \varphi}{1 - \Phi} \right\}. \quad (7.9)$$

The six second partial derivatives are

$$\frac{\partial^2 L}{\partial \beta_j \partial \beta_k} = \frac{\xi_i \xi_k}{\sigma^2} \left\{ -I + (1 - I) \right\}$$

$$\cdot \left\{ \frac{\xi_i \varphi}{1 - \Phi} - \frac{\varphi^2}{(1 - \Phi)^2} \right\}, \quad j, k = 0, 1,$$ 

$$\frac{\partial^2 L}{\partial \beta_j \partial \sigma} = -\frac{1}{\sigma} \frac{\partial L}{\partial \beta_j} + \frac{\xi_i}{\sigma} \left\{ -I 2 \xi_i + (1 - I) \right\}$$

$$\cdot \left\{ \frac{\xi_i \varphi}{1 - \Phi} - \frac{\varphi^2}{(1 - \Phi)^2} \right\}, \quad j = 0, 1, \quad (7.10)$$

These are given in terms of the random quantities $I$ and $z$ and the model parameters. The elements of the Fisher information matrix for an observation at $(\xi_i, \varphi)$ are the expectations

$$E\left\{ - \frac{\partial^2 L}{\partial \beta_j \partial \beta_k} \right\} = (\xi_i \xi_k/\sigma^2)$$

$$\cdot \left\{ 1 - \varphi \left( \xi - \frac{\varphi}{1 - \Phi} \right) \right\}, \quad j, k = 0, 1,$$ 

$$E\left\{ - \frac{\partial^2 L}{\partial \beta_j \partial \sigma} \right\} = (\xi_i/\sigma^2)$$

$$\cdot \left\{ 1 - \varphi \left( \xi - \frac{\varphi}{1 - \Phi} \right) \right\}, \quad j = 0, 1, \quad (7.11)$$

$$E\left\{ - \frac{\partial^2 L}{\partial \sigma^2} \right\} = (1/\sigma^4)$$

$$\cdot \left\{ 2 \Phi - \xi \varphi \left( 1 + \xi^2 - \frac{\xi \varphi}{1 - \Phi} \right) \right\}.$$ 

These expectations are calculated from (7.10) with the aid of the expectations

$$E[I] = \Phi, \quad E \frac{\partial L}{\partial \beta_j} = 0 (j = 0, 1), \quad \text{and} \quad E \frac{\partial L}{\partial \sigma} = 0;$$

the first of these expectations is a consequence of the definition of $I$, and the others are a general property of the first partial derivatives (7.9) evaluated at the true values of the parameters. Since $\varphi$ and $\Phi$ are functions of just $\xi$, the last three expressions in braces $\{\}$ are functions of $\xi$. Denote them by $A(\xi)$, $B(\xi)$, and $C(\xi)$, respectively. For $\xi_0 = 1$ the Fisher information matrix $F_1$, for an observation at $\xi$, has the form
The quantity \( r \) can be written in terms of the standardized quantities \( a \) and \( b \) as
\[
    r = \left( \frac{\eta - \beta_0}{\sigma} \right) - \left( \frac{\mu_0 - \mu_p}{\sigma} \right) \xi_i = a - b \xi_i. \tag{7.13}
\]
Thus the Fisher information matrix for an observation at \( t_i \) is a function of just \( a = (\eta - \beta_0)/\sigma \), \( b = (\mu_0 - \mu_p)/\sigma \), and \( \xi_i \).

The Fisher information matrix for any plan with a sample of \( n \) independent observations is
\[
    F = \sum_{i=1}^{n} F_{ik}, \tag{7.14}
\]
where the \( i \)th unit is tested at a \( \xi_i \) value of \( \xi_i \). For example, suppose there are \( K \) equally spaced (transformed) test stresses from \( \xi_i \) equals 0 to \( \xi \) and an equal allocation of \( 1/K \) of the \( n \) test units to each test stress. This is called a standard plan. Then the corresponding Fisher information matrix for the sample is
\[
    F = \left( \frac{n}{K} \right) F_0 + \sum_{i=1}^{K-1} F_{2i/(K-1)} + \cdots + F_{\frac{2(K-1)}{K}} \tag{7.15}
\]
where \( \xi_i = 0 \) corresponds to the highest test stress (with transformed value \( x_0 \)) and \( \xi \) corresponds to the lowest test stress, \( x_1 \). Here \( F \) is a function of \( a, b, \xi, \) and \( K \). Also, for another example, suppose there are two test stresses where a proportion \((1 - p)\) of the \( n \) test units is allocated to the high test stress \( \xi_i = 0 \) (\( x_0 \) is the transformed value), and the remaining proportion \( p \) is allocated to the low test stress \( \xi_i = k \) (\( x_1 \) is the transformed value). Then the corresponding Fisher information matrix for the sample is
\[
    F = (1 - p)NF_0 + pNF_1. \tag{7.16}
\]
This is a function of \( a, b, \xi, \) and \( p \). Nelson and Kiepinski (1972) provide an argument that the optimum plans use just two stresses.

**Variance of the Estimate of the Design Mean**

For any plan, the asymptotic covariance matrix \( \Sigma \) of the ML estimates \( \hat{\beta}_0 \), \( \hat{\beta}_1 \), and \( \delta \) is the inverse of the corresponding Fisher information matrix. That is,
\[
    \Sigma = \begin{bmatrix} \text{Var}(\hat{\beta}_0) & \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \text{Cov}(\hat{\beta}_0, \delta) \\ \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \text{Var}(\hat{\beta}_1) & \text{Cov}(\hat{\beta}_1, \delta) \\ \text{Cov}(\hat{\beta}_0, \delta) & \text{Cov}(\hat{\beta}_1, \delta) & \text{Var}(\delta) \end{bmatrix}^{-1}. \tag{7.17}
\]

The ML estimate of the 100\( P \)th percentile of the distribution at the design stress with reciprocal \( x_0 \) is
\[
    \hat{y}_P(1) = \hat{\beta}_0 + \hat{\beta}_1 \cdot t + z_P \delta \tag{7.18}
\]
where \( z_P \) is the 100\( P \)th standard normal percentile.

The asymptotic variance of the estimate is
\[
    \text{Var}(\hat{y}_P(1)) = \text{Var}(\hat{\beta}_0) + \text{Var}(\hat{\beta}_1) + 2 \text{Cov}(\hat{\beta}_0, \hat{\beta}_1). \tag{7.19}
\]

There is evidence that this asymptotic formula is generally satisfactory even for small samples. Harter and Moore [10], p. 212, used Monte Carlo methods to estimate these variances and covariances for \( n = 10 \) and 20 and Type II censoring of 0 to 90% of the sample in steps of 10%. They conclude “The precision of the maximum-likelihood estimator \( \hat{\mu} \), when proper allowance is made for bias, closely approximates that predicted by the asymptotic formula for the variance of \( \hat{\mu} \) ... except in cases of strong asymmetric censoring.” Examination of their tabulations of exact and asymptotic variances indicates that the asymptotic formulas are satisfactory for practical purposes if the censored fraction of the sample does not exceed 60% at each stress.

**Optimum Plans**

For a standard plan, the variance (7.20) is a function of \( a, b, \xi, \) and \( K \). Then \( \xi \) can be chosen to minimize this variance for given \( a, b, \) and \( K \) values. Similarly, for a plan with two stresses and unequal allocation, the variance (7.20) is a function of \( a, b, \xi, \) \( p \). Then \( \xi \) and \( p \) can be chosen to minimize this variance for a given \( a \) and \( b \) values.

The values of \( \xi \) and \( p \) that minimize the variance for standard and optimum plans can be numerically obtained as described by Nelson and Kiepinski [20]. The minimum variance for a sample of size \( n \) has the form
\[
    \text{Var}(\hat{y}_P(1)) = \sigma^2 V/n \tag{7.21}
\]
as may be seen from equations (7.14) through (7.17).

The variance factor \( V \) is a function of just \( a \) and \( b \); it is given by the charts for the standard and optimum plans presented by Nelson and Kiepinski [20]. The calculation of these factors is described by them. When an optimum plan is used in practice, the optimum number \( pm \) of test units at the low stress must be rounded to an integer. Consequently, the allocation may be one unit away from optimum.
Monte Carlo Verification of Theory

The asymptotic ML theory above applies to large samples. A Monte Carlo simulation was run to obtain the distribution of $\hat{\mu}_s$ for a sample size that would occur in practice. In particular, the simulation shows that the large-sample formula (5.2) is adequate. This suggests that the plans based on large-sample theory are satisfactory for sample sizes in practice.

The simulation involves the optimum plan for the example. For a sample size of $n = 40$, the plan calls for 11 units at 260°C and 29 at 159°C. The assumed parameter values of the model are $\alpha = -6.014$, $\beta = 4.3070$, and $\sigma = 0.2590$, which are the ML estimates obtained from the actual test data. Data for 50 such plans were simulated and analyzed to obtain the estimate $\hat{\mu}_s$ for the mean log life at 130°C. This was done with the STATPAC program of Nelson and Hendrickson [19]. Asymptotic theory says that the estimates $\hat{\mu}_s$ come from a cumulative distribution that is close to a normal one with mean $\mu_s = 4.672$ and standard deviation 0.1044 from (5.2).

The Monte Carlo estimates are plotted on normal probability paper in Figure 2. The straight line in the plot is the asymptotic normal distribution. The sample distribution and the asymptotic distribution agree quite well; Monte Carlo samples from a true normal distribution often differ more from the true cumulative distribution.

8. ACKNOWLEDGEMENT

This work received the generous support and encouragement of Dr. R. L. Shuey, Manager of the Information Studies Branch of General Electric Corporate Research and Development. This work was done under a cooperative program between GE Corp. Research and Development and Rensselaer Polytechnic Institute; this program benefited from the cooperation of Dr. John Wilkinson, Professor of Management and Chairman of the Interdisciplinary Program in Operations Research and Statistics. Mr. Delmar Crawford, formerly of the Small AC Motor Department, motivated the authors to do this work and kindly permitted the use of his data in the examples. Dr. Donald A. Gardiner, the Associate Editor, and a referee helped shorten this paper and

Figure 2—Normal Probability Plot of Monte Carlo Estimates
References


* Reprints and General Electric TIS Reports are available on request from the Distribution Unit, Bldg. 5, Room 237, GE Corp. Research & Development, Schenectady, N. Y. 12345.