Uniformly Most Powerful Unbiased Tests on the Scale Parameter of a Gamma Distribution With a Nuisance Shape Parameter

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Conditional tests on the scale parameter of the gamma distribution with an unknown nuisance shape parameter are considered. Such tests, based upon the conditional distribution of the sample mean $\bar{x}$ (or equivalently $W_n = \bar{x}/\tilde{x}$) given the geometric mean $\tilde{x}$, are uniformly most powerful unbiased tests. Percentage points of the conditional distribution are tabulated for small sample sizes and an asymptotic normal approximation is also obtained.

KEY WORDS

Gamma Distribution
Scale Parameter
Conditional Tests
Normal Approximation

1. INTRODUCTION

The gamma distribution, as defined by the density function

$$f(x) = x^{k-1}\exp(-x/\beta)/\beta^k\Gamma(k), \qquad x > 0,\ k > 0,\ \beta > 0$$

has been useful in many areas of statistics. The parameters k and β are respectively shape and scale parameters. In the areas of life-testing and reliability, the parameter $\theta = 1/\beta$ is also of interest. In the exponential case ($k = 1$), θ is the constant failure rate of the distribution. In the general case ($k > 0$), θ is the asymptotic value of the failure rate function as $x \to \infty$ (see Barlow and Proschan [2], p. 14).

Let $x_1, \cdots, x_n$ denote a random sample of size n from a gamma distributed population with both θ and k unknown. The gamma distribution is a member of the exponential family, and it follows that the arithmetic mean, $\bar{x} = \sum_{i=1}^n x_i/n$, and the geometric mean, $\tilde{x} = (\Pi_{i=1}^n x_i)^{1/n}$, are joint complete sufficient statistics.

Based on the statistic $W_n = \bar{x}/\tilde{x}$, it is possible to construct tests with optimum properties for k with θ an unknown nuisance parameter. References concerning such tests are included in papers by Bain and Engelhardt [1] or Glaser [8]. It would be desirable to also have tests for θ (or β) with k an unknown nuisance parameter. As noted by Mann et al. ([13], p. 263), no general method for drawing inferences concerning θ with k unknown has been developed. The usual approach is to attempt to determine a statistic, preferably a function of the sufficient statistics, whose distribution is independent of the nuisance parameter. This may be difficult if the nuisance parameter is not related to a location or scale parameter. For the gamma distribution it is helpful to consider conditional tests. In this case the conditional density of $\bar{x}$ given $\tilde{x}$ is also a member of the exponential family, and is independent of k. It also follows from Lehmann ([11], p. 136) that conditional tests for θ with k unknown can be constructed, and that these are uniformly most powerful unbiased (UMPU) tests. Thus, theoretically a solution to our problem exists. Practical implementation requires percentage points of the conditional distribution, which is extremely complicated. Both small sample and asymptotic percentage points have been tabulated and their use in inference procedures is discussed in the next section. The asymptotic derivations and computation of the percentage points are then discussed further in later sections.

2. UMPU TESTS FOR θ WITH k UNKNOWN

Let $W_n = \bar{x}/\tilde{x}$ and $G_n = \theta\tilde{x}$ and consider the conditional distribution function of W_n given $G_n = g$, written $F_{W_n}(w \mid g)$. The advantage of considering these variables is that the distribution depends on the

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two quantities w and g, and not separately on $\bar{x}$, $\tilde{x}$ and θ. It is also helpful to have the variable standardized relative to n and to g. This allows asymptotic values to be included in Table 1, and also regularizes the values to improve interpolation accuracy. Thus, Table 1 provides percentage points $u_\gamma(g)$ such that $P\{\sqrt{n}\, g[W_n - E(W_n \mid g)] \le u_\gamma(g) \mid g\} = \gamma$, for various values of n, g and γ. Values of $E(W_n \mid g)$ are also included in Table 1. For large g, it can be shown that $E(W_n \mid g) \to 1$ and $u_\gamma(g) \to u_\gamma(\infty) = \sqrt{n}[\chi_\gamma^2(n-1)/2n - (1 - 1/n)/2]$, where $\chi_\gamma^2(n-1)$ denotes the chi-square percentage point for $n - 1$ degrees of freedom. For large n, an asymptotic normal approximation is provided, which results in the asymptotic critical values $u_\gamma(g) = \sqrt{c_g}\, z_\gamma$, where z_γ denotes the normal percentage point and m_g and c_g denote asymptotic approximations for $E(W_n \mid g)$ and $ng^2\,\mathrm{Var}(W_n \mid g)$. Values of m_g and c_g are provided in Table 2.

These asymptotic results were used to provide the limiting values for large g and large n which permits interpolation on $1/n$ or $1/g$ for large values of g or n.

TABLE 1—*Values $E(W_n \mid g)$ and $u_\gamma(g)$ such that $P\{\sqrt{n}\, g[W_n - E(W_n \mid g)] \le u_\gamma(g) \mid g\} = \gamma$*

			γ								
n	g	$E(W_n \mid g)$	.01	.025	.05	.10	.50	.90	.95	.975	.99
5	0.5	1.691	-.716	-.680	-.637	-.570	-.129	.736	1.073	1.400	1.816
	0.7	1.510	-.740	-.703	-.658	-.588	-.131	.758	1.104	1.438	1.866
	1.0	1.367	-.761	-.722	-.676	-.604	-.134	.778	1.131	1.472	1.908
	2.0	1.191	-.791	-.751	-.703	-.627	-.138	.807	1.171	1.523	1.973
	5.0	1.078	-.812	-.771	-.721	-.644	-.141	.827	1.200	1.560	2.016
	20.0	1.020	-.822	-.780	-.730	-.651	-.141	.840	1.218	1.581	2.039
	∞	1.000	-.828	-.786	-.736	-.657	-.144	.845	1.227	1.597	2.074
10	0.5	1.777	-.946	-.863	-.779	-.664	-.093	.783	1.092	1.383	1.744
	0.7	1.575	-.979	-.893	-.805	-.686	-.095	.808	1.128	1.428	1.803
	1.0	1.414	-1.008	-.919	-.828	-.705	-.097	.829	1.156	1.464	1.847
	2.0	1.215	-1.046	-.954	-.859	-.732	-.098	.860	1.198	1.516	1.914
	5.0	1.088	-1.072	-.977	-.880	-.749	-.101	.882	1.228	1.554	1.959
	20.0	1.022	-1.087	-.990	-.892	-.759	-.103	.894	1.245	1.573	1.982
	∞	1.000	-1.093	-.996	-.897	-.764	-.104	.899	1.252	1.585	2.003
20	0.5	1.819	-1.100	-.977	-.860	-.712	-.065	.795	1.078	1.337	1.651
	0.7	1.607	-1.140	-1.013	-.892	-.738	-.067	.825	1.121	1.393	1.725
	1.0	1.438	-1.173	-1.042	-.917	-.759	-.069	.847	1.151	1.430	1.771
	2.0	1.228	-1.218	-1.082	-.952	-.788	-.072	.878	1.193	1.482	1.834
	5.0	1.093	-1.250	-1.110	-.977	-.809	-.074	.899	1.221	1.516	1.876
	20.0	1.024	-1.264	-1.123	-.988	-.817	-.074	.913	1.239	1.539	1.906
	∞	1.000	-1.271	-1.128	-.993	-.822	-.074	.917	1.246	1.549	1.922
30	0.5	1.833	-1.163	-1.022	-.891	-.729	-.052	.795	1.066	1.311	1.606
	0.7	1.618	-1.208	-1.061	-.926	-.757	-.055	.828	1.112	1.371	1.684
	1.0	1.446	-1.243	-1.092	-.952	-.779	-.057	.851	1.144	1.410	1.734
	2.0	1.232	-1.290	-1.133	-.988	-.808	-.058	.883	1.186	1.463	1.798
	5.0	1.095	-1.323	-1.162	-1.013	-.828	-.059	.905	1.216	1.498	1.841
	20.0	1.024	-1.340	-1.177	-1.026	-.839	-.060	.917	1.231	1.518	1.865
	∞	1.000	-1.346	-1.182	-1.031	-.843	-.060	.921	1.238	1.527	1.879
40	0.5	1.840	-1.200	-1.048	-.909	-.738	-.045	.795	1.059	1.297	1.583
	0.7	1.623	-1.247	-1.090	-.945	-.768	-.048	.828	1.106	1.357	1.660
	1.0	1.450	-1.284	-1.121	-.972	-.790	-.049	.852	1.138	1.396	1.708
	2.0	1.234	-1.331	-1.164	-1.009	-.820	-.051	.883	1.179	1.447	1.771
	5.0	1.096	-1.364	-1.192	-1.033	-.839	-.051	.906	1.209	1.483	1.815
	20.0	1.024	-1.382	-1.207	-1.047	-.850	-.052	.917	1.225	1.502	1.837
	∞	1.000	-1.389	-1.213	-1.052	-.854	-.053	.922	1.231	1.512	1.852
∞	0.5	1.866	-1.442	-1.215	-1.020	-.795	.000	.795	1.020	1.215	1.442
	0.7	1.640	-1.483	-1.250	-1.049	-.818	.000	.818	1.049	1.250	1.483
	1.0	1.462	-1.522	-1.283	-1.076	-.839	.000	.839	1.076	1.283	1.522
	2.0	1.240	-1.578	-1.330	-1.116	-.870	.000	.870	1.116	1.330	1.578
	5.0	1.098	-1.617	-1.363	-1.144	-.891	.000	.891	1.144	1.363	1.617
	20.0	1.025	-1.638	-1.380	-1.158	-.903	.000	.903	1.158	1.380	1.638
	∞	1.000	-1.645	-1.386	-1.163	-.907	.000	.907	1.163	1.386	1.645

The above results now make it possible to construct tests or confidence limits for θ with k unknown. For example, a UMPU size α test of $H_0:\theta \leq \theta_0$ against $H_a:\theta > \theta_0$ is to reject H_0 if

$$\sqrt{n}\, g_0[\bar{x}/\tilde{x} - E(W_n \mid g_0)] < u_\alpha(g_0),$$

where $g_0 = \theta_0\tilde{x}$.

The associated confidence limits can also be obtained, although they are less convenient than the testing situation.

To determine a $1 - \alpha$ lower confidence limit θ_L, first determine the value g_L which satisfies

$$\sqrt{n}\, g_L[\bar{x}/\tilde{x} - E(W_n \mid g_L)] = u_\alpha(g_L) \qquad (1)$$

then $\theta_L = g_L/\tilde{x}$.

The right hand side of equation (1) is nearly constant, and the left hand side increases with g_L so a solution for g_L can be obtained readily by trial and error.

The methods will be illustrated by application to some numerical examples.

EXAMPLE 1

Gross and Clark [9] considered the following random sample of 20 survival times (in weeks) of male mice exposed to 240 rads of gamma radiation.
152, 152, 115, 109, 137, 88, 94, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83, 69.

For this sample $\bar{x} = 113.5$, $\tilde{x} = 107.1$ and $w = \bar{x}/\tilde{x} = 1.06$. The maximum likelihood estimates (MLE's) are $\hat{k} = 8.74$ and $\hat{\theta} = .075$. Suppose we wish to test $H_0:\theta \leq .05$ against $H_a:\theta > .05$ at the .01 significance level. Then $g_0 = \theta_0\tilde{x} = .05(107.1) = 5.36$. By interpolation on $1/g$ in Table 1, we have $E(W_{20} \mid 5.36) \doteq 1.09$ and $u_{.01}(5.36) = -1.25$. Since $\sqrt{n}\, g_0[\bar{x}/\tilde{x} - E(W_n \mid g_0)] = \sqrt{20}(5.36)(1.06 - 1.09) = -.72$, H_0 cannot be rejected at the .01 level. Now suppose a .99 lower confidence limit for θ is desired. Consider equa-

TABLE 2—*Asymptotic Values* m_g and $c_g = ng^2v_g$

g	m_g	c_g
0.1	4.3859	0.2770
0.2	2.9078	0.3203
0.3	2.3523	0.3483
0.4	2.0541	0.3686
0.5	1.8660	0.3841
0.6	1.7360	0.3965
0.7	1.6404	0.4067
0.8	1.5670	0.4151
0.9	1.5089	0.4222
1.0	1.4616	0.4282
1.1	1.4225	0.4335
1.2	1.3895	0.4380
1.3	1.3613	0.4420
1.4	1.3369	0.4455
1.5	1.3156	0.4487
2.0	1.2398	0.4603
2.5	1.1934	0.4677
3.0	1.1621	0.4729
4.0	1.1224	0.4794
5.0	1.0983	0.4835
10.0	1.0496	0.4917
20.0	1.0249	0.4958
40.0	1.0125	0.4979
80.0	1.0062	0.4990
∞	1.0000	0.5000

tion (1) after dividing by $\sqrt{n}$. For an initial value of $g = .05\bar{x} \doteq 5$, $u_{.01}(5)/\sqrt{20} = -.280$. Values of $g_L[\bar{x}/\tilde{x} - E(W_n \mid g_L)]$ for $g_L = 5, 4, 3$ are respectively, $-.165$, $-.222$, $-.279$. Recomputing the right hand side gives $u_{.01}(3)/\sqrt{20} = -0.276$. This is probably as accurate as the data, but additional trials give $g = 3.04$ and $\theta_L = 3.04/107.1 = .028$.

EXAMPLE 2

Choi and Wette [7] provide the results of a simulated sample of size 200. The data, which was generated from a gamma distribution with $\theta = 1$ and $k = 3$, yield $\bar{x} = 2.905$ and $\tilde{x} = 2.455$. Suppose it is desired to test the hypothesis $H_0: \theta = .9$ against the alternative $H_a: \theta > .9$. Since $g_0 = (.9)(2.455) = 2.10$ we obtain by interpolation in Table 1, $E(W_{200} \mid 2.10) = 1.228$ and $u_{.05}(2.10) = -1.10$. Since $\sqrt{n}\, g_0[\bar{x}/\tilde{x} - E(W_n \mid g_0)] = \sqrt{200}(2.10)[1.183 - 1.228] = -1.34$, H_0 can be rejected at the .05 level. Suppose it is desired to construct a 95% confidence interval for θ. Following the approach outlined in *Example 1*, upper and lower 97.5% limits can be obtained. The resulting confidence interval would be (.86, 1.29). An alternate approach for large sample sizes would be to use the asymptotic normal approximation from Section 4 directly. For example, in the confidence interval problem, limits for an asymptotic level $1 - \alpha$ confidence interval can be obtained as $\theta_L = g_L/\tilde{x}$ and $\theta_U = g_U/\tilde{x}$ with $g = g_U$ and $g = g_L$ the respective solutions to $\pm z_{\alpha/2} = \sqrt{n}\, g[\bar{x}/\tilde{x} - m_g]/\sqrt{c_g}$ where $\bar{x}$ and $\tilde{x}$ are observed values. A convenient initial value for the asymptotic method is $g_0 = 1/2(\bar{x}/\tilde{x} - 1)$. For the numerical example, $g_0 = 1/2(1.183 - 1) = 2.7$, and the solutions to $\pm 1.960 = \sqrt{200}\, g[1.183 - m_g]/\sqrt{c_g}$ obtained by interpolation in Table 2 are $g_U = 3.18$ and $g_L = 2.11$. Thus, (2.11/2.455, 3.18/2.455) = (.86, 1.30) is an asymptotic 95% confidence interval for θ. It appears that the asymptotic normal approximation would have been adequate in this case.

It is also interesting to compare to the confidence interval based upon the asymptotic normal distribution of $\hat{\theta}$ with the variance estimated. Since $\hat{k} = 3.133$, $\hat{\theta} = 1.079$ and $\mathrm{Var}(\hat{\theta}) \doteq .0306$, the resulting confidence interval for θ is (.74, 1.42), which is wider than the confidence interval based on the optimal approach.

3. SMALL-SAMPLE RESULTS

It is possible to derive explicit expressions for the small-sample conditional distribution and moments of W_n given $G_n = g$, although the numerical evaluation of these expressions is quite involved. Since, for any specified k, $\tilde{x}$ is a complete sufficient statistic for θ and $L = \tilde{x}/\bar{x}$ is distributed independently of θ, it follows from the results of Basu (1955) that $\bar{x}$ and L are independent statistics. The statistic $\bar{x}$ is gamma distributed with mean k/θ and variance $k/\theta^2 n$ and the density function $p(l)$ of L was derived by Nair (1938). By a change of variables, the joint density function of $W_n = 1/L$ and G_n can be derived and the resulting conditional density function of W_n given $G_n = g$ is $f_{W_n}(w \mid g) = w^{nk-2} \exp(-ngw)p(1/w)/I(g, n, k)$, where $I(g, n, k) = \int_1^\infty t^{nk-2} \exp(-ngt)p(1/t)dt$. It was shown by Nair that $p(l) = F_1(l)F_2(l)$ where $F_1(l) = c(n, k)l^{nk-1}$ and $F_2(l)$ does not depend on k. Hence, $f_{W_n}(w \mid g) = (1/w)\exp(-ngw)F_2(1/w)/I(g, n)$, $w > 1$, where $I(g, n) = \int_1^\infty (1/t)\exp(-ngt)F_2(1/t)dt$. The conditional distribution function $F_{W_n}(w \mid g)$ could then be obtained by numerically integrating $f_{W_n}(w \mid g)$. The major difficulty involves the numerical evaluation of the function $F_2(l)$. Nair obtained a series solution, whose complexity increases with the sample size. Hartley (1940) used a chi-square series to evaluate the distribution of the variable $-2nk \ln L$. Of course, with a change of variables, this would also provide a series evaluation of $p(l)$. A solution with a similar form was also obtained by Glaser [8] by a different approach. Box [6] proposed a generalization of Hartley's chi-square series which provides an excellent method of evaluating $p(l)$ with a relatively small amount of computation. The small-sample results in Table 1 were based on this method.

The limiting results for large g were obtained as follows: The conditional mean can be expressed as

$$E(W_n \mid g) = \int_1^\infty w \exp(-ngw) w^{-1} F_2(w^{-1})\, dw \Big/ \int_1^\infty \exp(-ngw) w^{-1} F_2(w^{-1})\, dw.$$

After the substitution $t = ng(w - 1)$ and simplification we have $E(W_n \mid g) = 1 + (1/ng)\, Q_1(ng)/Q_0(ng)$ where

$$Q_i(z) = \int_0^\infty t^i \exp(-t)(-t)(1 + t/z)^{-1} F_2([1 + t/z]^{-1})\, dt.$$

It can be shown that

$$Q_i(z) \sim \int_0^\infty t^i \exp(-t)(t/z)^{(n-3)/2}\, dt = z^{-(n-3)/2}\Gamma(i + [n - 1]/2) \text{ as } z \to \infty.$$

Thus, for any fixed n, $ng[E(W_n \mid g) - 1] \sim \Gamma(1 + [n - 1]/2)/\Gamma([n - 1]/2) = (n - 1)/2$ as $g \to \infty$. It follows that $\lim_{g\to\infty} E(W_n \mid g) = 1$. By a similar argument we can show that $u_\gamma(\infty) = \lim_{g\to\infty} u_\gamma(g) = \sqrt{n}[\chi_\gamma^2(n - 1)/2n - (1 - 1/n)/2]$. The limiting values for large g in Table 1 were based on these results.

4. ASYMPTOTIC RESULTS

Following the notation of Section 2, let $G_n = \theta\tilde{x}$ and $W_n = \bar{x}/\tilde{x}$. Furthermore, define $m_g = \psi^{-1}(\ln g)/g$ for each $g > 0$, where ψ^{-1} denotes the inverse of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$, and let $Y_n =$

$\sqrt{n}[W_n - m_{G_n}] = \sqrt{n}[W_n - \psi^{-1}(\ln G_n)/G_n]$. If $\{a_n\}$ and $\{b_n\}$ are numerical sequences, the notation $a_n \sim b_n$ will have the usual meaning that $\lim_{n\to\infty} (a_n/b_n) = 1$.

Theorem: Let $F_{Y_n}(y)$ denote the distribution function of Y_n and define $F(y \mid g) = \Phi(yg/[\psi^{-1}(\ln g) - 1/\psi'(\psi^{-1}(\ln g))]^{1/2})$ where $\Phi(z)$ denotes the standard normal distribution function. Then, for each real number y,

$$E_{G_n}[F(y \mid G_n)] \sim F_{Y_n}(y) \quad (2)$$

Furthermore, $F(y \mid g)$ is a distribution function in the variable y, a continuous function of g, independent of k and n, and the only such function which satisfies property (2).

Proof: By relating W_n and G_n to the MLE's we can find the limiting distribution of Y_n. It is well known that the MLE's $\hat{k}$ and $\hat{\theta}$ have an asymptotic bivariate normal distribution with asymptotic means k and θ respectively. The asymptotic variances and covariance, as given by Choi and Wette (1969), are Var $(\hat{k}) = k/D$, $\text{Var}(\hat{\theta}) = \theta^2\psi'(k)/D$, and $\text{Cov}(\hat{k}, \hat{\theta}) = \theta/D$ where $D = n[k\psi'(k) - 1]$. It is easily verified that $W_n = \hat{k} \exp[-\psi(\hat{k})]$ and $G_n = (\theta/\hat{\theta}) \exp[\psi(\hat{k})]$. In the Lemma of Rao (1952, p. 207) let $T_1 = \hat{k}$, $T_2 = \hat{\theta}/\theta$ and $f(t_1, t_2) = [t_1 - t_2\psi^{-1}(\psi(t_1) - \ln t_2)] \exp[-\psi(t_1)]$. It follows that the limiting distribution of Y_n is normal with mean 0 and variance $[k - 1/\psi'(k)] \exp[-2\psi(k)]$, so that $\lim_{n\to\infty} F_{Y_n}(y) = F(y) = \Phi(y \exp[\psi(k)]/[k - 1/\psi'(k)]^{1/2})$. Now, since G_n is asymptotically normal it converges stochastically to the asymptotic mean $\exp[\psi(k)]$. Furthermore, since $F(y \mid g)$ is a bounded, continuous function of g, it follows from the Helly-Bray Theorem (see Loeve, [12], p. 182) that $\lim_{n\to\infty} E_{G_n}[F(y \mid G_n)] = F(y \mid \exp[\psi(k)]) = F(y)$. Since $E_{G_n}[F(y \mid G_n)]$ and $F_{Y_n}(y)$ have the same finite, nonzero limit, this verifies property (2). Suppose $H(y \mid g)$ is another distribution function in the variable y which is continuous in g, independent of k and n, which satisfies (2). Since $H(y \mid g)$ is a bounded continuous function of g, it follows by the Helly-Bray Theorem and property (2) that $\lim_{n\to\infty} E_{G_n}[H(y \mid G_n)] = H(y \mid \exp[\psi(k)])$ for any $k > 0$. This implies that $H(y \mid g) = F(y \mid g)$ for any $g > 0$. The fact that $F(y \mid g)$ is a distribution function in the variable y and a continuous function of g follows easily from basic properties of the functions involved. This completes the proof.

Let $F_{Y_n}(y \mid g)$ denote the conditional distribution function of Y_n given $G_n = g$. An important property relating the functions $F_{Y_n}(y)$ and $F_{Y_n}(y \mid g)$ is that

$$E_{G_n}[F_{Y_n}(y \mid G_n)] = F_{Y_n}(y) \quad (3)$$

In fact, by completeness the function $F_{Y_n}(y \mid g)$ is determined by this relationship. The derivation of $F_{Y_n}(y \mid g)$ as a solution of (3) is not tractable. However, if we view expression (2) as an asymptotic analog of (3), then the Theorem provides us with an asymptotic solution, namely $F(y \mid g)$. In this sense, the asymptotic conditional distribution of W_n given $G_n = g$ could be regarded as normal with mean $m_g = \psi^{-1}(\ln g)/g$ and variance $v_g = n^{-1}[\psi^{-1}(\ln g) - 1/\psi'(\psi^{-1}(\ln g))]/g^2 = (1/ng^2)c_g$. The asymptotic quantities m_g and c_g are provided in Table 2. These results have also been used to provide the asymptotic values of m_g and $u_\gamma(g) = z_\gamma\sqrt{c_g}$.

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