Properties of the Mixed Exponential Failure Process

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Various properties of the mixed exponential hazard rate and reliability are discussed. Particular attention is given to the relationship between the mixed hazard rate h(t) and the mixing density g(\lambda). Characteristic functions are derived and expressions relating integral transforms (Laplace, Fourier and Mellin) to the mixed exponential family are presented. A complex contour integration is used to invert a given mixed R(t) and recover the mixing density g(\lambda). Several generalized mixed reliability functions and the corresponding mixing densities are provided together with their Muth global classification measures. The relationship of the mixed exponential hazard rate to the intensity function of some generalized stochastic point processes is discussed. Results for a mixed Weibull density are also included.

KEY WORDS
Mixed exponential reliability
Mixing densities
Integral transforms
Special functions
Mixed Weibull reliability

1. INTRODUCTION

When the distribution function of the random time-to-failure for any component is exponential the hazard rate is known to be a constant, say \lambda. But in a population of such components there may be an ubiquitous variation in \lambda-values because of small fluctuations in manufacturing tolerances, so that a component selected at random can be regarded as having a random hazard rate, call it A. Here A takes values with the appropriate frequency in the set of \lambda-values. The distribution of A across the positive real numbers is called the mixing distribution (or density).

In Harris and Singpurwalla (1968) and McNolty (1964) a gamma mixing density was employed for A as a useful description of the aforementioned manufacturing variability. If A has a gamma distribution, with shape parameter \alpha and scale parameter \beta both positive, then the density is given by

\[ g(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} \quad \text{for} \ \lambda > 0. \] (1.1)

Here \( E(\Lambda) = \alpha \beta, \ Var(\Lambda) = \alpha \beta^2 \) and if \( \alpha > 1 \) then the mode is \( \beta(\alpha - 1) \). For later convenience, we define its reciprocal

\[ m = 1/\beta(\alpha - 1). \] (1.2)

The time-to-failure, say T, of a particular component with hazard rate mixed according to the random variable \Lambda will have a reliability \( R(t) \) given by

\[ R(t) = E[\exp(-\Lambda t)] = \int_0^\infty e^{-\lambda} g(\lambda) d\lambda - (1 + \beta t)^{-\alpha} \] for \( t \geq 0 \) (1.3)

and density \( f \) given by

\[ f(t) = -R'(t) = \alpha \beta (1 + \beta t)^{-\alpha-1} \] for \( t \geq 0; \] (1.4)

see Harris and Singpurwalla (1968). The mixed exponential hazard rate, by definition, becomes

\[ h(t) = f(t)/R(t) = \alpha \beta (1 + \beta t)^{-\alpha} \] for \( t \geq 0. \] (1.5)

One sees this hazard rate is decreasing since \( h'(t) < 0 \), but such a property can be shown to be true for any mixture by a straightforward application of Schwartz's inequality. See Barlow, Marshall and Proschan (1963).

The concepts underlying these formulas may be clarified by referring directly to Harris and Singpurwalla (1968), and Mann, Schafer and Singpurwalla (1974).

The Muth (1977) measure \( m_G \) for global memory corresponding to (1.3) is given by

\[ m_G = \begin{cases} \int_0^\alpha \frac{-\infty}{1 - \alpha} & \text{for } 1 \leq \alpha \leq 2 \\ \int_0^\infty \frac{-\infty}{1 - \alpha} & \text{for } \alpha > 2 \end{cases} \] (1.6)

It is informative to introduce a rather unusual graphical display which relates the mixing density to the mixed hazard rate. In Figure 1 the time axis extends to the right of the origin as usual, while the horizontal
axis to the left of the origin represents the ordinates of the pdf \( g(\lambda) \) for \( \alpha > 1 \). The vertical axis represents units of \( \lambda \) and also represents the amplitude of \( h(t) \). Since \( \lambda \) is measured in units of \( t^{-1} \) the ordinates of \( h(t) \) are expressible in terms of a rate. One might even think of the vertical axis as denoting frequency in which case \( h(t) \) may loosely be interpreted as a "low-pass filter." Admittedly the frequency analog is not perfect since we are mainly interested in the first and only failure. In this case there would not be a period between failures nor a frequency (cps) of occurrence of failures. Nor would there be a rate of occurrence of failures, for that matter. Nevertheless the terminology has a certain intuitive convenience.

Beginning with \( t = 0 \) the hazard rate takes on values which decrease in "frequency" from \( \alpha \beta \) to zero. Here \( \alpha \) is a dimensionless quantity while \( \beta \) has units of \( t^{-1} \). The mixed hazard rate \( h(t) \) corresponds to the first and only failure of a randomly selected component from the population described by \( g(\lambda) \). At a time \( t_m \), Figure 1 shows that \( h(t) \) takes on the modal value \( \beta(\alpha - 1) \) and that the indicated rectangular area equals unity.

In an ensemble sense \( h(t) \) also represents the instantaneous average \( \lambda \) of all remaining components, i.e., those which have not yet failed. Accordingly at \( t = 0 \) the hazard rate properly assumes the value \( \alpha \beta = E(\lambda) \) and as \( t \) becomes increasingly large \( h(t) \) takes on smaller and smaller values. Figure 2 illustrates the manner in which the gamma distribution, corresponding to the remaining components, changes as \( t \) increases.

![FIGURE 1. Frequency interpretation of the mixed exponential failure process when the mixing density has the gamma form with \( \alpha > 1 \).](image)

![FIGURE 2. Evolution of the gamma mixing density corresponding to the remaining (unfailed) components.](image)
A change in the shape parameter of the mixing density from $\alpha$ to $K\sqrt{\alpha}$ results in a new hazard rate which is translated upward.

$$h_K(t) = h(t) + K \cdot \sigma_a(1 + \beta t)^{-1}. \quad (1.7)$$

Also we note that

$$h_K(t_m) = \beta(\alpha - 1)(1 + K/\sqrt{\alpha}) = h(t_m)(1 + K/\sqrt{\alpha}) \quad (1.8)$$

where $h_K(t_m) = h(t_m)$ for large $\alpha$ and again $t_m$ is the time at which $h(t)$ assumes the modal value $\beta(\alpha - 1)$.

Figure 3 relates the shape parameter $\alpha$ of $g(\lambda)$ to the mixed hazard rate $h(t)$. In terms of the previous references to "frequency", the small values of $\alpha$ engender greater "low-pass filter" characteristics in $h(t)$. The case $\alpha < 1$, for instance, shows the heavy weighting of the very low frequencies where the frequencies in the lower bandwidth translate, via $1/\lambda$, into very large mean times to failure.

### 2. CHARACTERISTIC FUNCTIONS

The characteristic function $\phi(u)$ for the mixed exponential pdf (1.6) is given by

$$\phi(u) = \int_{-\infty}^{\infty} e^{iu \lambda} \, \hat{g}(\lambda) \, d\lambda = \alpha \beta \int_{0}^{\infty} e^{iu \lambda}(1 + \beta \lambda)^{-\alpha - 1} \, d\lambda$$

$$= [\beta/\Gamma(\alpha)]^{-1} \int_{0}^{\infty} v^{-\alpha} e^{-v(\beta + iu)} \, dv$$

$$\quad \cdot \exp(iu/\beta) \Gamma(-\alpha, u/\beta) \quad (2.1)$$

where

$$\Gamma(-\alpha, u/\beta) = \int_{u/\beta}^{\infty} e^{-x} x^{-\alpha - 1} \, dx \quad \text{and} \quad i - \sqrt{-1}. \quad (2.4)$$

One checks easily that

$$\lim_{u \to 0} \phi(u) = 1 \quad (2.2)$$

as must be the case.

Also in a purely formal manner (from McNolty, 1964), without regard to convergence we may write

$$\phi(u) = \alpha \sum_{k=0}^{\infty} (iu)^{\alpha K} \Gamma(\alpha - K)[\beta^{K} \Gamma(\alpha + 1)]^{-1} \quad (2.3)$$

where expression (2.3) is a useful artifice for obtaining the moments of $f(t)$ in a simple manner. Thus, for $\alpha > 1$ the mean time to the first failure becomes

$$E(T) = t_m = \frac{1}{\alpha - 1} = \frac{1}{\alpha} \quad (2.4)$$

and for $\alpha > 2$ the variance for the time to first failure is

$$\text{Var}(T) = \frac{1}{\alpha} \cdot \frac{1}{\alpha - 2} = \frac{1}{\alpha - 2} \quad (2.5)$$

When the argument of the mixed pdf is $t_m$ we have

$$f(t_m) = \beta(\alpha - 1) \cdot \left(\frac{\alpha - 1}{\alpha}\right)^{t_m} \cdot h(t_m) \quad (2.6)$$
Expressions such as (2.4) can also be calculated in a direct manner through integration by parts, e.g.,

\[
E(T) = \beta \int_0^\infty \frac{t \, dt}{(1 + \beta t)^{x+1}}
\]

\[
= \frac{\alpha}{\tilde{\beta}} \int_1^{\infty} \frac{du}{u^x - \tilde{\beta}} = \frac{\alpha}{\frac{\alpha - 1}{\beta}} - 1/\beta
\]

\[
= 1/\beta(x - 1) - 1/\beta
\]

\[
= 1/\beta(x - 1), \quad \alpha > 1. \quad (2.7)
\]

3. INTEGRAL TRANSFORM RELATIONSHIPS

Integral transforms have been studied extensively in terms of their structure and applications; thus, it is often fruitful to formulate problems in the transform format. For the most part, transform notation is standardized; however, in order to introduce some functional notation which is peculiar to our discussion it is worthwhile to briefly review those transforms of interest in this section. These linear operators will then be related to the mixed exponential functions.

Laplace transform of \( f(x) = L[f(x); s] \)

\[
= L_f(s) = \int_0^\infty f(x) e^{-sx} \, dx. \quad (3.1)
\]

Mellin transform of \( f(x) = M[f(x); s] \)

\[
= M_f(s) = \int_0^\infty f(x) x^{s-1} \, dx. \quad (3.2)
\]

Fourier transform of \( f(x) = F[f(x); s] \)

\[
= F_f(s) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx. \quad (3.3)
\]

Characteristic function of \( f(x) = C[f(x); s] \)

\[
= \theta_f(s) = \int_{-\infty}^{\infty} f(x) e^{itx} \, dx. \quad (3.4)
\]

The inversions of (3.1)–(3.4) are, respectively,

\[
f(x) = (1/2\pi i) \int_{-\infty}^{\infty} L_f(s) e^{sx} \, ds = L^{-1}[L_f(s); x]
\]

where \( s = u + iv \), (3.5)

\[
f(x) = (1/2\pi i) \int_{-\infty}^{\infty} M_f(s) x^{-s} \, ds = M^{-1}[M_f(s); x]
\]

where \( s = u + iv \), (3.6)

\[
f(x) = (1/2\pi) \int_{-\infty}^{\infty} F_f(s) e^{i\omega x} \, ds = F^{-1}[F_f(s); x]
\]

and

\[
f(x) = (1/2\pi) \int_{-\infty}^{\infty} \theta_f(s) e^{-itx} \, ds = \theta_f^{-1}(x). \quad (3.7)
\]

There is, of course, no intrinsic difference between (3.3) and (3.4); however, since the notation of (3.4) is quite well established in the probability literature, it is included here together with the most prevalent form (3.3) of the Fourier transform found in the electrical engineering literature. Many European authors, incidentally, prefer to multiply the integral of (3.1) by \( s \) and then invert \( L_f(s)/s \).

With each passing year the engineering and statistical literature includes more frequent applications of the Mellin transform and for this reason it has been included briefly in this section.

In all cases \( f(x) \) is assumed to be a continuous function for which the transform exists and similarly for the inversions. In (3.5), for instance, \( L_f(s) \) is assumed to be an analytic function of the complex variable \( s = u + iv \) over the half plane \( u \geq 0 \) and it is also assumed to be of order \( O(s^{-K}) \), for \( K > 1 \), in this region. If, in addition, \( L_f(u) \) is real for \( u \geq 0 \) then the integral (3.5) converges to the real valued function \( f(x) \) which is independent of \( c \) where \( c \geq 0 \). In the case of (3.6) one considers a strip of analyticity rather than a half plane. In all cases one must select a permissible range for each of the parameters so that convergence is assured. See (3.39b), for instance.

One can now write some useful and interesting expressions for the mixed exponential reliability \( R(t) \), hazard rate \( h(t) \) and pdf \( f(t) \). The relationships which follow apply to any continuous mixing density \( g(t) \) for which the integrals converge, except for the Mellin transform where it is assumed that \( g(t) \) has the gamma form (1.1).

The Laplace transform relationships are as follows:

\[
h(t) = \frac{R(t)}{R(\infty) - t} = \frac{Q(A)}{(A); t} \quad (3.9)
\]

so that \( h(t) \) may be expressed as the ratio of two Laplace transforms

\[
h(t) = L[[g(\lambda); t] / L[g(\lambda); t] = L[y(t) / L_g(t). \quad (3.10)
\]

Also we have

\[
E[\Lambda / (\Lambda + s)] = E[L[z(t) / \Lambda]; s] = L[f(t); s] \quad (3.11)
\]

where \( E(\cdot) \) is the expected value notation and \( r(t) \) is the exponential density for the random time to failure for a particular component as described in the Introduction, i.e.,

\[
r(t) = \lambda e^{-\lambda t} \quad (3.11a)
\]

where the corresponding component reliability is, of course,

\[
r(t) = e^{-\lambda t} \quad (3.11b)
\]

Continuing further, the Fourier transform and characteristic function relationships are:

\[
R(t) = \theta_f^{(-t)} = F_f^{(-t)} \quad (3.12)
\]

\[
f(t) = \theta_g^{(-t)} = F_g^{(-t)} \quad (3.13)
\]
which yields
\[ h(t) = F[\lambda g(\lambda); -it]/F[g(\lambda); -it] \]
\[ = F[\lambda g(0)/-it]/F[g(\lambda); -it] \]  \hspace{1cm} (3.14)

and
\[ h(t) = C[\lambda g(\lambda); it]/C[g(\lambda); it] \]
\[ = \theta_A(it)/\theta_B(it). \]  \hspace{1cm} (3.15)

When the mixing pdf \( g(\lambda) \) has the gamma form (1.1)
the Mellin transform gives the following expressions
\[ f(t) = M[\lambda w(\lambda, t); \alpha] = M_{\lambda w}(\alpha, t) \]  \hspace{1cm} (3.16)
\[ R(t) = M[w(\lambda, t); \alpha] = M_w(\alpha, t) \]  \hspace{1cm} (3.17)

and
\[ h(t) = M_{\lambda w}(\alpha, t)/M_w(\alpha, t) \]  \hspace{1cm} (3.18)

where
\[ w(\lambda, t) = [\Gamma(\alpha)\beta^\alpha]^{-1} \cdot \exp[-\lambda(t + 1/\beta)]. \]  \hspace{1cm} (3.19)

A simpler version of (3.18) may be obtained by first translating (3.11a) and (3.11b) in time by an amount
\( 1/\beta \) (i.e., \( t \rightarrow t + 1/\beta \)) and denoting the resulting functions by \( \tau f(t) \) and \( r f(t) \), respectively. Then
\[ h(t) = M_{\tau f}(\alpha, t)/M_{r f}(\alpha, t). \]  \hspace{1cm} (3.20)

In order to provide an application of integral transform methods to the mixed exponential functions, we will now assume a given functional form for \( R(t) \) and work backwards to obtain the corresponding mixing density \( g(\lambda) \). This reverses the procedure in which one always selects \( g(\lambda) \) and then determines \( R(t) \) or \( f(t) \).

Here we are interested in reliability functions which have the following properties:
(a) \( R(t) \) is always positive for \( t \geq 0 \),
(b) \( R(0) = 1 \),
(c) \( R(t) \) is a monotonically decreasing function of time,
(d) \( R(t) \) meets the integrability conditions for inversion. (Recall the discussion following (3.8)).

Referring to the Laplace inversion we now have
\[ g(\lambda) = (1/2\pi i) \int_{c_1 - i\infty}^{c_1 + i\infty} e^{\lambda \tau} R(\tau) d\tau \]  \hspace{1cm} (3.21)

and for this example \( R(t) \) will be chosen so as to yield a mixing pdf of the form (1.1). Thus, we let
\[ R(t) = (1 + t/b)^{-a} \]  \hspace{1cm} (3.22)

where \( a, b > 0 \).

From (3.21)
\[ g(\lambda) = (1/2\pi i) \int_{-\infty}^{\infty} e^{\lambda z}(1 + z/b)^{-a} dz \]  \hspace{1cm} (3.23)

and letting \( z = ix \) in (3.23) gives
\[ g(\lambda) = (b^a/2\pi) \int_{-\infty}^{\infty} e^{ixz}(b + ix)^{-a} dx \]  \hspace{1cm} (3.24)

so that we consider the contour integration
\[ \int_{C} e^{izx}(b + ix)^{-a} dx = \int_{C_1} + \int_{C_2} + \cdots + \int_{C_6} = 0 \]  \hspace{1cm} (3.25)

where \( C \) and the paths \( C_1, C_2, \ldots, C_6 \) are shown in Figure 4.

Evaluating each of the integrals individually, one can show that:
\[ \int_{C_1} e^{ix(b + ix)^{-a}} dx = 0 \] for \( a > 0 \) and \( \lambda > 0 \) \hspace{1cm} (3.26)

\[ \lim_{R \to \infty} \int_{C_2} = 0. \] \hspace{1cm} (3.27)

FIGURE 4 Contour for the inversion of a complex reliability function \( R(z) \) to obtain the corresponding mixing density \( g(\lambda) \).
Having shown that the integrals on C2 and C6 vanish as \( R \to \infty \), we are now free to make a convenient transformation. We let

\[
    w = b + iz = b - y + ix = u + iv.
\]

That is, the old real x-axis now becomes the imaginary axis in the complex w-plane and the old imaginary iy axis becomes the axis of reals. In (3.25) this substitution yields

\[
    \int_{C2} e^{iz(b + iz)^{-a}} \, dz = -ie^{-ib} \int_{C6} w^{-a} \cdot e^{i\omega} \, dw
\]

from which one can show that

\[
    \lim_{\rho \to 0} \int_{C_3} = i(-\pi - \pi)K
\]

where

\[
    K = \lim_{w \to 0} w \cdot \left[ -i e^{-a} \cdot e^{i(w-b)} \right] = ie^{-ib} \lim_{w \to 0} w^{i-1} = 0
\]

and (3.1) holds for \( a < 1 \). The previous restriction on the parameter "a" in (3.27) was \( a > 0 \). Strictly speaking, we must impose the condition \( 0 < a < 1 \) because of (3.1); however, it turns out that "a" may be extended throughout the range \( a > 0 \) by means of analytic continuation.

Considering the segments \( C_3 \) and \( C_5 \) it is noted that

\[
    \int_{C_2} + \int_{C_3} \neq 0,
\]

because \( C_3 \) and \( C_5 \) are on opposite sides of a branch cut. It may be shown that

\[
    \lim_{\rho \to 0} \int_{C_5} = ie^{-ib} \cdot e^{i\pi(1-a)} \cdot \lambda^{-a-1} \cdot \Gamma(1-a)
\]

and

\[
    \lim_{\rho \to 0} \int_{C_5} = -e^{-ib} \cdot e^{-i\pi(1-a)} \lambda^{-a-1} \Gamma(1-a).
\]

Thus, as \( R \to \infty \) and \( \rho \to 0 \)

\[
    \int_{C} e^{iz(b + iz)^{-a}} \, dz
\]

\[
    = \int_{-\infty}^{\infty} e^{ix(b + ix)^{-a}} \, dx + i\pi \left[ e^{i(1-a)} - e^{-i(1-a)} \right] = 0.
\]

Finally, by simplifying (3.35) and transposing terms,

\[
    \int_{-\infty}^{\infty} e^{i\pi(b + ix)^{-a}} \, dx = \left[ 2\pi \Gamma(a) \right] \lambda^{-a-1} e^{-ib}.
\]

From (3.24) and (3.36) the mixing distribution is seen to be

\[
    g(\lambda) = \left( b^a / 2\pi \right) \cdot \left[ 2\pi \Gamma(a) \right] \lambda^{-1} e^{-ib}
\]

\[
    = \left[ b^a \Gamma(a) \right] \cdot \lambda^{-a-1} e^{-ib}
\]

which may be written in the form (1.1) simply by letting \( a = \alpha \) and \( b = 1/\beta \).

The preceding example has shown that we can select a desired reliability \( R(t) \) from a permissible class of functions and work backwards through the complex inversion integral to obtain the corresponding mixing distribution \( g(\lambda) \).

Another example of inverting the reliability function is obtained by letting \( R(t) \) take the form

\[
    R(t) = a^b [(a + b) (b + t)]^{-1}; \quad a, b > 0
\]

and from McNolty (1973) the inversion (3.21) yields

\[
    g(\lambda) \, d\lambda = a^b \Gamma(\mu + \lambda) \cdot \exp(-b\lambda) \cdot \lambda^{a-1} \cdot \text{I}_1 F_1(\nu, \mu; \lambda) \, d\lambda, \quad \lambda \geq 0
\]

where

\[
    \nu + \mu > 0; \quad a > 0 ; \quad b > 0
\]

and \( \text{I}_1 F_1(a; c; x) \) is the confluent hypergeometric function (Whittaker and Watson, 1927). Alternatively, \( g(\lambda) \) may be written (McNolty, 1973) in the form of a mixture representation

\[
    g(\lambda) \, d\lambda = \sum_{a=0}^{\infty} NB(n; \nu; a/b) \cdot G(\lambda; b; \mu + v + n) \, d\lambda
\]

where \( NB(n; \nu; a/b) \) is the discrete negative binomial distribution

\[
    NB(n; \nu; a/b) = \left( \frac{a^n}{n^n} \right) \cdot \left( \frac{a}{a-b} \right)^n
\]

and \( G(\lambda; b; \mu + v + n) \) is the continuous gamma density defined in (1.1) with \( \beta = 1/b > 0 \) and \( \alpha = \mu + v + n \).

In this case the decreasing hazard rate has the form

\[
    h(t) = \sum_{n=0}^{\infty} NB(n; \nu; a/b) \cdot (\mu + v + n) b^{-n-1} (a + t)^{n+1} (\beta + t)^{-1}
\]

where the pdf \( f(t) \) for the time to failure is given by

\[
    f(t) = h(t) \cdot a^b [(a + t)(b + t)]^{-1} \, dt
\]

where, alternatively, (3.43) may be written as

\[
    f(t) = \sum_{n=0}^{\infty} NB(n; \nu; a/b) \cdot (\mu + v + n) b^{-n-1} (a + t)^{n+1}
\]

Finally, by simplifying (3.35) and transposing terms,

\[
    \int_{-\infty}^{\infty} e^{i\pi(b + ix)^{-a}} \, dx = \left[ 2\pi \Gamma(a) \right] \lambda^{-a-1} e^{-ib}.
\]
or as a special-function representation
\[ f(t) = a^b b^v (v + \mu) (b + t)^{-u - v - 1} \]
\[ \cdot \, _2 F_1 [v, \mu + v + 1; v + \mu; (b - a)/(b + t)]. \] (3.45)

The expected value of \( \lambda \) with respect to \( g(\lambda) \) may be expressed in terms of \( f(t) \) as
\[ E(\Lambda) = f(0) = (a\mu + bv)/ab. \] (3.46)

The expression (3.42) for the hazard rate may also be written in the alternative form,
\[ h(t) = (\mu + v)(a + t)^{-u - v - 1} \]
\[ \cdot \, _2 F_1 [v, v + \mu + 1; v + \mu; (b - a)/(b + t)]. \] (3.47)
where \( _2 F_1(a, b; c; x) \) is the hypergeometric function (Whittaker and Watson, 1927).

A referee suggests a direct method for obtaining \( g(\lambda) \) from (3.38) which may be applied when the permissible range of the parameters in (3.39b) is restricted further. The referee’s method is discussed in the Appendix and provides additional insight into the probabilistic behavior of the mixing density \( g(\lambda) \).

The Muth (1977) measure \( m_G \) for global memory corresponding to (3.38) is given by the following expression
\[ m_G = 2 - (2/i^2) \cdot a^b b^v (b - a)^{1 - v - u} \]
\[ \cdot \left[ \frac{[b - u] \cdot B_{1 - ab}(\mu + v - 2, 1 - v)}{b \cdot B_{1 - ab}(\mu + v - 1, 1 - v)} \right] \] (3.48)
where \( B_{a,b}(x, y) \) is the incomplete beta function and \( \overline{t} \) is the mean time to failure,
\[ \overline{t} = E(T) = a^b b^v (b - a)^{1 - v - u} \]
\[ \cdot \, B_{1 - ab}(\mu + v - 1, 1 - v), \quad b > a. \] (3.49)

In the derivation of (3.48) we have imposed the convergence condition \( v + \mu > 1 \). Also, if \( a < b \) we simply interchange \( a \) and \( b \) and interchange \( v \) and \( u \) in (3.48).

Another example is provided by defining \( R(t) \) as
\[ R(t) = \alpha^p (\alpha + t)^{-P} \cdot \exp[-\beta^2 t/4a(\alpha + t)] \] (3.50)
where \( P > 0 \) and \( \alpha, \beta > 0 \).

The inversion integral (3.21) now yields the mixing density \( g(\lambda) \) in the form
\[ g(\lambda) d\lambda = (2/P)^{p-1} \cdot \lambda^{(P-1)/2} \]
\[ \cdot \alpha^P \cdot e^{-2\lambda} \cdot \exp(-\beta^2/4a) \]
\[ \cdot \, I_{P-1}(\beta^2/\lambda) \, d\lambda, \quad \lambda \geq 0 \] (3.51)
where \( I_p(x) \) is the modified Bessel function of the first kind of order \( v \) (Watson, 1944).

Expression (3.51) may also be written in the form of a mixture representation
\[ g(\lambda) d\lambda = \sum_{m=0}^{\infty} Q(m; \beta^2/4a) \cdot G(\lambda; \alpha, P + m) \] (3.52)
where \( Q(m; a) \) is the familiar discrete Poisson distribution and \( G(x; a, b) \) is again the continuous gamma density.

The hazard rate for this case becomes
\[ h(t) = P/(\alpha + t + \beta^2/4(\alpha + t)^2 \] (3.53)
and the pdf for time to failure is
\[ f(t) dt = P\alpha^p \cdot (\alpha + t)^{-P-1} \cdot \exp(-\beta^2/4a) \]
\[ \cdot \, _1 F_1 [P + 1; P; \beta^2/4(\alpha + t)] \, dt \]
\[ = \sum_{m=0}^{\infty} Q(m; \beta^2/4a) \cdot (P + m) \alpha^{P+m} \]
\[ \cdot \, (\alpha + t)^{-P-m-1} \, dt \]
\[ = \alpha^P (\alpha + t)^{-P-1} \cdot \left[ P + \beta^2/4(\alpha + t) \right] \]
\[ \cdot \ Crexp[-\beta^2 t/4a(\alpha + t)] \, dt \] (3.54)
where
\[ E(\Lambda) = f(0) = P/\alpha + \beta^2/4a^2 \] (3.55)

The Muth \( m_G \) measure is somewhat difficult to obtain for the mixed \( R(t) \) given by (3.50) and because of various convergence conditions, we limit the following result to the range \( P > 3 \) with \( \alpha \geq 0 \) and \( \beta \geq 0 \). We obtain
\[ m_G = 2 - 2(P^2/4a)^{P-1} \cdot \exp(\beta^2/4a) \]
\[ \cdot \left[ F(P - 2, P; \beta^2/4a) \right]^{-2} \]
\[ \cdot \, \left[ (\beta^2/4a)^{P-3, \beta^2/4a} \right] \]
\[ \cdot \, \exp(-\beta^2 t/4a(\alpha + t)) \] (3.56)
where
\[ F(a, b) = \int_0^b x^a e^x \, dx \]
and we have required that \( a > 0 \) in our derivation of (3.56). In this case the mean time to failure \( \overline{t} \) is given by
\[ \overline{t} = E(T) = a^P (2/\beta^2)^{P-2} \cdot \exp(-\beta^2/4a) \]
\[ \cdot \, F(P - 2, P; \beta^2/4a) \] (3.57)
where (3.56) and (3.57) correspond to the mixed family of functions (3.50)–(3.55).

4. THE MIXED WEIBULL DISTRIBUTION
Harris and Singpurwalla (1968) treat several prior (mixing) and parent distributions where the Weibull is included in the latter category. In the very brief discussion in this section, we will essentially augment
their developments by considering a slightly different functional form for the Weibull density function. That is, analogous to (3.11a) and (3.11b),

\[
\tau(t) \, dt = \Pr[\text{time-to-failure } T \text{ occurs in } (t, t + dt)] = \lambda(t)^{t - 1} \cdot \exp[-(\lambda t)^{y}] \, dt, \quad t \geq 0
\]

\[
= 0 \quad \text{for } t < 0; \quad \lambda, \gamma > 0
\]

(4.1)

\[
r(t) = \text{reliability} = \Pr(\text{time-to-failure } T > t) = \exp[-(\lambda t)^{y}]
\]

(4.2)

\[
\theta(t) \, dt = \text{hazard rate}
\]

\[
\tau(t) / r(t) = \lambda(t)^{y - 1}.
\]

(4.3)

First it is assumed that the scale parameter is a random variable \( \Lambda \) distributed according to the gamma mixing density function (1.1). This yields a mixed (unconditional) pdf for the time-to-failure \( T \) given by

\[
f(t) = \gamma \beta^y \Gamma^{y-1} \sum_{K=0}^{\infty} (-1)^K (\beta t)^K \Gamma(\alpha + \gamma + K) / K!
\]

and similarly the mixed reliability becomes

\[
R(t) = [\Gamma(\alpha)]^{y} \sum_{K=0}^{\infty} (-1)^K (\beta t)^K \Gamma(\alpha + \gamma + K) / K!
\]

(4.4)

(4.5)

In this case the unconditional hazard rate \( h(t) \), being the ratio of the two series (4.4) and (4.5), does not have a simple form as in expression (1.5).

One can show that the \( m \)th moment of the time-to-failure \( T \) is given by

\[
E(T^m) = \int_0^\infty t^m f(t) \, dt = \Gamma(\gamma/m + 1) \Gamma(\alpha - m) / \Gamma(\alpha)
\]

(4.6)

where only those moments exist for which \( m < \alpha \).

5. STOCHASTIC POINT PROCESSES

Mixed exponential hazard rates such as (1.5) and (3.53) refer to the first failure, since they are inherently properties of a distribution. These expressions will now be used to define the intensity function \( \lambda_\phi(t) \) for a pure birth process which will correspond to multiple failures. Roughly speaking, the single failure phenomenon described by \( h(t) \) is to be propagated into a countable infinity of failures over the real time axis.

A. In the first case \( \lambda_\phi(t) \) is defined so that

\[
\lambda_\phi(t) = h(t) = \alpha \beta/(1 + \beta t)
\]

and

\[
\lambda_\phi(t) = h(t) + \eta/(1 + \beta t)
\]

\[
= (\alpha \beta + \eta)/(1 + \beta t), \quad n \geq 1
\]

(5.1)

where \( \lambda_\phi(t) \) is the intensity function defined Patil and Boswell (1970) as a collection \( N(t) \) of nonnegative, integer-valued random variables. The quantity \( N(t) \) is the population size at time \( t \) and satisfies, as \( h \to 0 \) from the right, \( \Pr[N(t + h) = n + 1 | N(t) = n] = \lambda_\phi(t) + o(h) \).

Pr[N(t + h) = n | N(t) = n] = 1 - \lambda_\phi(t) \cdot h + o(h); \) the probability of more than one birth during \( (t, t + h) \) is \( o(h) \). As usual, \( P_n(t) \) is the probability of \( n \) births (failures) during the time interval \((0, t)\) where \( n = 0, 1, \ldots; P_n(0) = 1 \) and \( P_{-K}(t) = 0 \) for any positive integer \( K \). Then one can show that

\[
P_n(t) = \left( -\alpha \beta/\gamma \right)_n \left[ 1 - (1 + \beta t)^{-\gamma/\beta} \right] \cdot (1 + \beta t)^{-\gamma/\beta}
\]

(5.2)

where (5.2) is a negative binomial process with parameters

\[
p = (1 + \beta t)^{-\gamma/\beta}, \quad r = \alpha \beta/\gamma
\]

and

\[
q = (\gamma/\alpha)P_1(t)/P_0(t) = 1 - p.
\]

When \( n = 0 \) expression (5.2) yields \( P_0(t) = R(t) = (1.7) \).

Let \( \beta = \gamma = a = 1/\alpha \); then (5.2) reduces to the Polya process

\[
P_n(t) = 1 \cdot (1 + \alpha)/(1 + 2\alpha) \cdots [1 + (n - 1)\alpha]
\]

\[
\cdot r(1 + \alpha t)^{-n} / n!
\]

(5.3)

where (5.3) has an intensity given by

\[
\lambda_a(t) = (1 + an)/(1 + at).
\]

B. In the second case the intensity function is defined (from equation (3.53)) by

\[
\lambda_a(t) = P/(a + t) + \beta^2/4(a + t)^2;
\]

\[
\lambda_n(t) = [(n + P)/(a + t)]
\]

\[
\cdot \Gamma(1 + P + 1; P; x)/\Gamma(1 + P; x)
\]

(5.4)

\[
\cdot F_1(n + P + 1; P; x/n)
\]

where \( F_1(a; c; x) \) is the confluent hypergeometric function and with \( x = \beta^2/4(a + t) \), (5.4) reduces to (3.55) for \( n = 0 \). It can be shown that

\[
P_n(t) = \alpha \beta^2 \Gamma(n + P) \cdot \exp(-\beta^2/4a)
\]

\[
\cdot [n! (a + t) \cdot \Gamma(T/P)]^{-1}
\]

\[
\cdot F_1(n + P; P; \beta^2/4(a + t))
\]

(5.5)

where \( P_0(0) = 1; \quad P_n(0) = 0 \) for \( n \neq 0 \) and \( P_n(t) = R(t) = (3.50) \).
A somewhat more complicated case arises when \( \lambda_d(t) \) is defined as
\[
\lambda_d(t) = (\alpha \beta + \gamma n^2)/(1 + \beta t)
\]
where (from equation (1.5))
\[
\lambda_d(t) = h(t) = \alpha \beta/(1 + \beta t)
\]
which leads to
\[
P_d(t) = -P_d(t)(\alpha \beta + \gamma n^2)/(1 + \beta t)
+ P_{n-1}(t) \left[ (\alpha \beta + \gamma(n-1)^2)/(1 + \beta t) \right]
\]
and
\[
\gamma S^2(S - 1) \cdot \frac{\partial^2 P(S, t)}{\partial S^2} + \gamma S(S - 1) \cdot \frac{\partial P(S, t)}{\partial S}
- (1 + \beta t) \cdot \frac{\partial P(S, t)}{\partial t} + \alpha \beta(S - 1)P(S, t)
= \gamma S(S - 1)P_1(t),
\]
where \( P_1(t) \) is obtained by setting \( n = 1 \) in (5.2). Expressions (5.7) and (5.8) have solutions
\[
P_n(t) = (\alpha \beta/\gamma) \cdot x^n \cdot \frac{\Gamma(n + i \nu)\Gamma(n - i \nu)}{\Gamma(1 + i \nu)\Gamma(1 - i \nu)}^{-1}
\cdot \left[ \frac{(n!)^2}{[(n - K)!]^2} + 2 \sum_{k=1}^{n} (-1)^k x^{-2k/(n)} \cdot \left(\frac{k}{n} - k\right)^{-1} \right]
\]
where \( x = 1 + \beta t, i = \sqrt{-1}, v = \sqrt{\alpha \beta/\gamma} \) and the sum over \( K \) is omitted when \( K = 0 \),
\[
P(S, t) = 2\alpha \beta x^{-x} \cdot \frac{\Gamma(1 + i \nu)\Gamma(1 - i \nu)}{\Gamma(n + i \nu)\Gamma(n - i \nu)}^{-1}
\cdot \sum_{n=0}^{\infty} \epsilon_n \cdot \frac{\Gamma(2n + 1)}{\Gamma(n + 1)}
\cdot x^{-2n/(n)} \cdot \frac{\Gamma(n + i \nu)\Gamma(n - i \nu)}{\Gamma(n + i \nu - 1)} \cdot 2F_1(i \nu, -i \nu; 2n + 1; S)
\]
where \( x, i, v \) are as defined in (5.9); \( \epsilon_n \) is equal to \( 1 \) for \( n = 0 \) and \( 1/2 \) for \( n > 0 \) and \( 2F_1(a, b; c; S) \) is the hypergeometric function.

For instance, from (5.9) one can write (as in equation (1.3)):
\[
P_d(t) = (1 + \beta t)^{-x}
\]
\[
P_1(t) = (\alpha \beta/\gamma) \cdot \left[ (1 + \beta t)^{\nu} - 1 \right] \cdot (1 + \beta t)^{-x + \nu/\beta}
\]
\[
P_2(t) = \alpha \beta(\alpha \beta + \gamma) t^{-2} \cdot P_d(t) \left[ (1 + \beta t)^{\nu} - \frac{3}{2}(1 + \beta t)^{-\nu/\beta}
+ \frac{1}{2}(1 + \beta t)^{-4\nu/\beta} \right]
\]
and so on.

Aside from the requirement that \( \lambda_d(t) = h(t) \), as given by (1.5) and (3.53), the selection of \( \lambda_d(t) \) is quite arbitrary. In (5.1) and (5.4), however, we have made the choice unique by imposing the condition that each member of the ensemble \( A \) (recall the Introduction) must correspond to a homogeneous Poisson process.

6. REMARKS

Interested readers are referred to the careful discussion of stochastic hazard rates, manufacturing variability and other ramifications of this topic which are provided in Harris and Singpurwalla (1968). The general concept is important in reliability theory and applications; however, the notion of a stochastic hazard is not limited to failure phenomena alone. In Section 5 above, for instance, we have in effect discussed doubly stochastic Poisson point processes which are known to characterize photoelectron emissions. In infrared sensor applications the intensity function associated with the Poisson distributed photon noise may be expressed as a function of such parameters as the incident irradiance \( H \) at the sensor, aperture diameter \( D \), optical efficiency \( \eta_r \), sampling time \( t \) and photon energy \( hc/\lambda \). In turn, \( H \) is determined by the radiant emissivity \( e(\lambda) \) of the target, the target’s projected area \( A \), the slant range \( R \) to the target from the sensor, the temperature \( T \) of the target’s surface in degrees Kelvin, etc. In some studies it is eminently reasonable to assign a mixing \( g(\lambda) \) to one of these parameters such as \( A \), for instance, whence \( g(A) \) is transferred to the intensity \( \lambda \) through a constant factor. Thus, the results of Section 5 can be applied directly to detection problems involving fluctuating targets or targets of unknown size.

Additional lucid discussions of this general subject are provided by Harris (1967, 1968) and Mann, Schafer and Singpurwalla (1974).

7. SUMMARY AND CONCLUSIONS

The analysis presented in this paper has examined some important relationships which exist between the unconditional reliability functions (hazard rate, reliability and pdf) and the corresponding mixing densities. These relationships have been interpreted in terms of frequency, integral transforms and special functions. Additionally the mixing densities and unconditional distributions have themselves been studied from the perspective of their characteristic functions, probabilistic mixture representations [such as (3.40), (3.44), (3.52) and (3.54)], special function representations [such as (3.39), (3.45), (3.47), (3.51) and (3.54)] and the Muth global measure [(1.6), (3.48) and (3.56)]. Lastly the various mixed (unconditional) hazard rates have been used in a manner analogous to an initial condition to define time and state \( n \) dependent intensity functions corresponding to several pure birth processes including a generalized...
Polya process, a confluent hypergeometric process and a process which is quadratic in the state variable \( n \). Figures 1 and 3 show the influence of the shape parameter \( \alpha \) on the "low-pass filter" characteristics of the mixing process which become more pronounced as \( \alpha \) becomes smaller. Figure 2 illustrates an ensemble interpretation of the evolution of the mixing pdf \( g(\lambda) \) with increasing time as the remaining components have smaller and smaller average \( \lambda \) values. Figure 2 is supported by the Muth global measure \( m_G \) of (4.6) which shows that the mixed failure process has a negative memory. The characteristic function (c.f.) for the gamma mixed exponential pdf has been derived (2.1) together with a divergent form (2.3) of the c.f. which may conveniently be used to obtain the moments of \( f(t) \) in a simple manner.

8 Appendix

An alternative method for inverting the mixed reliability \( R(t) \) to recover the mixing density \( g(\lambda) \) has been provided by a referee. For instance let \( R(t) \) be given by (3.42):

\[
R(t) = a' b' [(a + t)/(b + t)]^{-\mu}; \quad a, b > 0.
\]

Then since

\[
R_1(t) = E[\exp(-\Lambda t)] = (1 + \beta t)^{-\mu}
\]

when \( \Lambda \) is distributed according to (1.1), \( g(\lambda) = g(\lambda; \alpha, \beta) \), it is seen that

\[
R(t) = E[\exp(-\Lambda_1 t - \Lambda_2 t)] = E[\exp(-\Lambda_1 t)]
\]

where \( \Lambda_1 \) is distributed according to \( g(\lambda; v, 1/a) \) and the independent \( \Lambda_2 \) is distributed according to \( g(\lambda; \mu, 1/b) \). Thus the resulting mixing pdf can be extracted from the convolution of two gamma variates, i.e.,

\[
g(\lambda) = g(\lambda; v, 1/a) * g(\lambda; \mu, 1/b)
\]

where the necessary integration must be performed and coagulated into some recognizable form such as (3.43) or (3.44). The expectation (3.50) is now obtained via

\[
E(\Lambda_1 + \Lambda_2) = v/a + \mu/b
\]

and, of course, the mixing pdf corresponds to a random variable \( \Lambda \) which has the resolution

\[
\Lambda = \Lambda_1 + \Lambda_2.
\]

The permissible range of the parameter values attendant to this method is

\[
a, b > 0 \quad \text{and} \quad v, \mu > 0
\]

which is more restrictive than (3.43b).

In the case of (3.52) the referee writes

\[
R(t) = (1 + t/\alpha)^{-\mu} \cdot \exp[-vt/(\alpha + t)].
\]

by letting \( v = \beta^2/4\alpha \) and then effectively introduces a fourth parameter by writing

\[
R(t) = (1 + t/\alpha)^{-\mu} \cdot \exp[-vt/(\alpha + t)].
\]

One then considers a sequence of independent identically distributed random variables

\[ Y_1 + Y_2 + \cdots + Y_N \]

where \( N \) is random and distributed according to

\[
\Pr(N = n) = v^n e^{-v/n!} = P(n)
\]

which has the well-known probability generating function

\[
P(S) = E(S^N) = \exp[-v(1 - S)].
\]

The independent variates \( Y_i \) are each distributed according to

\[
g_i(y_i) \, dy = a \exp(-ay_i) \, dy
\]

where

\[
E[\exp(-Y_i t)] = a/(a + t).
\]

Thus

\[
E[\exp(-Y_i t)] = (1 + t/\alpha)^{-\mu}
\]

Finally, letting \( \Lambda \) be distributed according to (1.1), \( g(\lambda; P, 1/a) \), where

\[
E[\exp(-\Lambda t)] = (1 + t/\alpha)^{-\mu}
\]

it is seen that the resulting mixing pdf has the interesting resolution

\[
\Lambda + Y_1 + Y_2 + \cdots + Y_N.
\]

Again the actual mixing pdf must be obtained via integration and assembled into some convenient form such as (3.53). Here all parameters must be positive.

References


