Orthogonal Resolution-IV Designs for Some Asymmetrical Factorials

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Orthogonal resolution-IV designs for $4^2 \cdot 2^{2n-4}$, $4^3 \cdot 2^{4n-12}$, and $8^2 \cdot 2^{4n-8}$ experiments are constructed, where $n$ is a multiple of four. The plans for $4^3 \cdot 2^{4n-12}$ and $8^2 \cdot 2^{4n-8}$ experiments are minimal as well.

KEY WORDS: Resolution-IV designs; Hadamard matrix; Minimal designs.

1. INTRODUCTION

A fractional factorial plan is said to be of resolution-IV if it permits estimation of all main effects in the presence of two-factor interactions when three-factor and higher-order interactions are assumed to be zero. Margolin (1969) obtained a lower bound for the number of runs required for any resolution-IV design (orthogonal or not). If there are $i$ factors at $i$ levels ($i = 2, 3, \ldots, k$), then Margolin proved that

\[ N \geq k \left[ \sum_{i=2}^{k} (i-1)i_t - (k-2) \right], \tag{1.1} \]

where $N$ denotes the number of runs required for a resolution-IV design. Designs attaining the lower bound (1.1) are called minimal. A class of orthogonal minimal resolution-IV designs for $u \cdot 2^u$ experiments, $u$ being an even integer, was reported by Margolin (1969). A few orthogonal resolution-IV designs can also be obtained by applying the collapsing procedure (see, e.g., Addelman 1972). Near minimal resolution-IV designs for asymmetrical factorials were recently reported by Anderson and Thomas (1979). However, these plans are nonorthogonal in nature.

The purpose of this article is to present orthogonal resolution-IV designs for the following asymmetrical factorials:

1. $4^2 \cdot 2^{2n-4}$ in $8n$ runs,
2. $4^3 \cdot 2^{4n-12}$ in $16n$ runs, and
3. $8^2 \cdot 2^{4n-8}$ in $32n$ runs,

where $n$ is a multiple of four. The plans for $4^2 \cdot 2^{2n-4}$ and $8^2 \cdot 2^{4n-8}$ experiments are minimal, as the number of runs in these plans attains the lower bound (1.1), while the plans for $4^3 \cdot 2^{4n-12}$ are not minimal. However, it is shown that the number of runs required by this series of plans cannot be reduced further if orthogonal estimates are required.

Hadamard matrices were used to construct resolution-III plans by Addelman (1962) and more recently, by Dey and Ramakrishna (1977), Chacko et al. (1979), and Chacko and Dey (1981). In this article, the technique followed in those references for the construction of resolution-III plans is modified to yield resolution-IV plans for asymmetrical factorials.

2. PLANS FOR $4^2 \cdot 2^{2n-4}$ EXPERIMENTS

Let $n = 4t$, where $t$ is a positive integer. It is known that a Hadamard matrix of order $n$ exists for all $n \leq 200$ (Raghavarao 1971, p. 313; Turyn 1973). Hadamard matrices for all $n \leq 100, n \neq 92$, are displayed in Plackett and Burman (1946). The Hadamard matrix of order 92 has been constructed by Baumert et al. (1962). For further details on Hadamard matrices, the reader is referred to Hedayat and Wallis (1978).

Let $H$ be a Hadamard matrix of order $n$ and let, without any loss of generality, the first column of $H$ contain all unities. Obtain a matrix $B$ of order $n \times (n-1)$ from $H$ by deleting the first column of $H$. Rearrange the rows of $B$, so that in the first column of $B$, the first $n/2$ elements are $-1$ and the last $n/2$ elements are $1$ and call the new matrix $B^*$. Partition $B^*$ as $[b_1 : b_2 : B_3]$, where $b_1$ and $b_2$ are the first two columns of $B^*$ and $B_3$ is the $n \times (n-3)$ matrix of remaining columns in $B^*$. Further, let $J$ denote the $n$-component column vector of all unities.
Let \( B_1 = [b_3 : J] \), and define

\[ D_1^* = \begin{bmatrix}
    b_1 & b_2 & B_1 & -B_1 \\
    3b_1 & b_2 & B_1 & B_1 \\
    b_1 & 3b_2 & B_1 & B_1 \\
    3b_1 & 3b_2 & B_1 & B_1
\end{bmatrix}. \]

Consider the plan \( D_1 \), given by

\[ D_1 = \begin{bmatrix}
    D_1^* \\
    -D_1^*
\end{bmatrix}. \]

The matrix \( D_1 \) is a plan for a \( 4^2 \cdot 2^{2n-4} \) experiment in \( 8n \) runs, rows of \( D_1 \) representing the runs. The first two factors in \( D_1 \) are at four levels each, the levels being codes as \(-3, -1, 1, 3\), while the remaining \((2n - 4)\) factors are each at two levels, \(-1\) and \(1\). It can be seen that in an \( 8n \times 3 \) submatrix of \( D_1 \), if two columns out of the three are for those factors that are at four levels, each of the 32 ordered triplets of levels occurs precisely \( t \) times. Again, if one of the columns out of three represents a four-level factor and the other two represent two-level factors, each of the 16 ordered triplets of the levels appears in the \( 8n \times 3 \) submatrix precisely \( 2t \) times. Finally, since the two-level part of \( D_1 \) is essentially a "foldover" plan, it follows that \( D_1 \) is an orthogonal array with variable symbols (Rao 1973) and is of strength three. Hence the plan \( D_1 \) is an orthogonal resolution-IV plan for \( 4^2 \cdot 2^{2n-4} \) experiment in \( 8n \) runs. Obviously, in this case, the lower bound (1.1) is \( 8n \); thus the plan is minimal also. By collapsing one of the 4-level factors to a 3-level factor, we can get a plan for a \( 4^3 \cdot 3^{n_1} \cdot 2^{2n-4} \) experiment in \( 8n \) runs, which permits the orthogonal estimates of all main effects. These plans, however, are not minimal.

3. PLANS FOR \( 4^3 \cdot 2^{4n-12} \) EXPERIMENTS

Let \( R^* \) be defined as earlier and partition \( R^* \) as

\[ B^* = [b_1 : b_2 : b_3 : B_4]. \]

where \( b_i (i = 1, 2, 3) \) represents the first three columns of \( B^* \) and \( B_4 \) is the \( n \times (n - 4) \) matrix of remaining columns in \( B^* \).

Let \( B_4^* = [B_4 : J] \) and define the matrix \( D_3^* \) given in Figure 1. The plan \( D_3^* \) given by

\[ D_3 = \begin{bmatrix}
    D_3^* \\
    -D_3^*
\end{bmatrix} \]

is proposed for a \( 4^3 \cdot 2^{4n-12} \) experiment in \( 16n \) runs.

As before, it can be shown that \( D_3 \) is an orthogonal array with variable symbols and strength three; hence \( D_3 \) is an orthogonal resolution-IV plan for \( 4^3 \cdot 2^{4n-12} \) experiment in \( 16n \) runs. The lower bound in this case turns out to be \((16n - 20)\), and as such the plans given by \( D_3 \) are not minimal in the sense of Margolin (1969). However, it can easily be seen that in order that \( D_3 \) be an orthogonal array with variable symbols of strength three, the minimum number of runs required is \( 16n \). Thus, in this sense, we may possibly regard the plan \( D_3 \) also as minimal orthogonal.
4. PLANS FOR $8^2 \cdot 2^{4a-8}$ EXPERIMENTS

The ideas presented in the previous sections can be generalized to obtain orthogonal resolution-IV designs for $8^2 \cdot 2^{4a-8}$ experiments in $32n$ runs.

Let, as before, $B_1 = [B_3 : J]$, and let $D_3^*$ be the matrix given in Figure 2.

Then consider the plan $D_3$ given by

$$D_3 = \begin{bmatrix} D_3^* \\ -D_3^* \end{bmatrix}.$$ 

$D_3$ is a plan for an $8^2 \cdot 2^{4a-8}$ experiment in $32n$ runs. Arguing as before, it can be shown that $D_3$ is an orthogonal resolution-IV plan for an $8^2 \cdot 2^{4a-8}$ experiment. The plan is minimal as well, as the lower bound (1.1) is precisely $32n$.

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REFERENCES


