TECHNOMETRICS, AUGUST 1989, VOL. 31, NO. 3

Comparing Renewal Processes, With Application to Reliability Modeling of Highly Reliable Systems

M. Tortorella

AT&T Bell Laboratories
Whippany, NJ 07981

Elementary methods are used to obtain upper and lower bounds for the difference between two renewal functions corresponding to two different life distributions and to obtain various order relationships between them that follow from order relationships on the underlying life distributions. The bounds are used to approximate the value of an unknown renewal function by the value of a known renewal function and to bound the error of approximation by expressions involving the difference between the two underlying life distributions. They are useful mainly in the region where time is small compared with the means of the two distributions, and they therefore supplement the information that can be obtained for large time using the elementary renewal theorem. These results find major application in reliability modeling of highly reliable systems.

KEY WORDS: Approximation; Distance; Error bounds; Order; Stochastic comparison.

1. INTRODUCTION

In this article, elementary methods are used to obtain easily computable upper and lower bounds for the difference between two renewal functions corresponding to two different life distributions. I also obtain various order and stochastic-order results for the corresponding renewal-counting processes, thereby making it possible to compare numbers of renewals in terms of the hazard rates of the underlying life distributions. The bounds are used to approximate the value of an unknown or difficult-to-compute renewal function by the value of a known or easier-to-compute renewal function and to bound the error of approximation by expressions involving the difference between the two underlying life distributions.

A renewal process (Thompson 1981, sec. 3) is a sequence $\{X_1, X_2, \ldots\}$ of mutually iid nonnegative random variables. In reliability-modeling applications, these random variables represent the successive lifetimes of a unit or system that, upon failure, is replaced by a new one or is overhauled to as-new condition. The renewal process is frequently used as a model for the reliability of a maintained system in which repair restores the system to as-new condition and repair times are negligible in comparison to operating times. Reliability studies often are interested in the number of failures of the unit over a given time interval. To obtain this in the renewal model, the renewal counting process $\{N(t) : t \geq 0\}$ is introduced. $N(t)$ is the number of failures that have occurred by time t. In the notation introduced previously, $N(t) = \max\{n : X_1 + \cdots + X_n \leq t\}$. The renewal function $M(t) = EN(t)$ is the expected number of failures that have occurred by time t and is an important reliability figure of merit for maintained systems in which the renewal model of repair is appropriate.

The limiting behavior of the renewal function for large time is well known, and I review this in Section 2.5. For highly reliable systems, however, it may be that the system design life is quite short compared with the time required for the limiting results to become valid as approximations to the transient (near time 0) values of the renewal function. Since renewal functions are available in closed form only in a few special cases, their transient behavior can be hard to obtain. Thus there is interest in obtaining bounds and approximations that would help pin down the expected number of system failures even for highly reliable systems and for small times.

I give such bounds in this article. They are tight mainly in the region where time is small compared to the means of the two distributions concerned, and therefore they supplement information that can be obtained for large time from the elementary renewal theorem (see Sec. 2.5). Indeed, the bounds I give are generally not very good for large time. In practical terms, this means that they are most useful for modeling highly reliable systems (in which the time it takes for the system's reliability process to reach steady state, or become approximately stationary, is large compared to the system design life) when it is also important to have a good measure of the expected number of system failures because of, say, great expense or inconvenience of repair. They are also useful for analysis of warranties when the warranty period may be considerably shorter than the mean of the time to the first system failure. The development of these results was motivated by the reliability analysis of an intercontinental undersea cable telecommunications system, in which use of the large-time approximations afforded by the elementary renewal theorem led to large errors. Repair on this system is accomplished by replacing a failed repairable unit (a repeater) by a new one, an expensive, disruptive, and time-consuming process. A renewal model was used for each repeater, because the effect of a repair was to install a new repeater in place of a failed one. Since a reliability objective for the whole system included a requirement on the number of system failures over its design life, a system-reliability model was chosen in which the renewal function for each repeater was multiplied by the number of repeaters in the system (all repeaters were identical). The system's individual components were so highly reliable, however, that its design life was comparable to the mean of the time to the first system failure and was short compared to the time it would take for the large-time limits to become useful as reliability figures of merit. The large-time limits from the elementary renewal theorem then led to overly pessimistic values for the expected number of repeater replacements over the system design life.

The bounds are useful in reliability computations and comparisons for maintained systems or units whose sequence of failures is modeled as a renewal process. Renewal functions are known in closed form only in a few special cases. In general, finding the renewal function for a life distribution involves summing an infinite series of convolution integrals (Cleroux and McConalogue 1976; McConalogue 1981), numerically solving the renewal integral equation (Deligonul and Bilgen 1984; Soland 1969; Tortorella 1987), or numerically inverting a Laplace transform. If hardware or software for performing these complicated numerical computations is not available, or if one simply wants to obtain an approximation for the reliability of a system modeled by a renewal process, one could replace the life distribution for the system by a distribution having a closed-form renewal function, and use the bounds given here to measure the error arising from this approximation procedure.

Closed forms are known for the renewal functions of the shifted exponential and truncated exponential distributions (Barlow and Proschan 1965, p. 57) and of the distributions of phase type. A distribution is of phase type if it is the distribution of time to absorption in a finite-state Markov chain having one absorbing state and all other states transient. Distributions of phase type were introduced by Neuts in 1975; all pertinent information can be found in Neuts (1981). Since the phase-type distributions are dense in the set of all continuous distributions in the uniform norm, any continuous distribution can be approximated by a phase-type distribution for which the supremum of the absolute difference between the two distributions is as small as desired (and specified at the beginning of the procedure). One could then choose to approximate the renewal function for the original distribution, whose renewal function may not be expressible in closed form, by the renewal function for the approximating phase-type distribution (which is computable). The results in this article can then be used to express the error of approximation for the renewal functions in terms of the error of approximation for the distributions.

The familiar exponential and Erlang (gamma with integer shape parameter) distributions are special cases of phase-type distributions. The renewal function for the exponential distribution $1 - \exp(-\lambda t)$ is well known as simply λt. The renewal functions for the Erlang distributions were given by Parzen (1962, chap. 5, eq. 2.30). For the general distribution of phase type, the renewal function was given by Neuts (1981, eq. 2.4.6), although computation of these renewal functions can still be a formidable task, since one needs to solve a (possibly large) system of ordinary differential equations (Neuts 1981, eq. 2.4.8) to get started. An algorithm for carrying out these computations was given by Kao (1988). Thus the exponential and Erlang distributions will be useful as comparison distributions when it is undesirable or impossible to employ extensive numerical computational facilities. A possible drawback is that they are underdispersed (coefficient of variation ≤ 1), and so probably would work best for systems whose life distributions had the same property.

For overdispersed distributions, suitable comparison distributions include mixtures of exponentials. The renewal function for the mixture $p(1 - e^{-\lambda t}) + q(1 - e^{-\mu t})$, $p + q = 1$, $t \geq 0$, and $\lambda, \mu > 0$, may be shown by elementary Laplace transform methods to be

$$\frac{\lambda\mu}{q\lambda + p\mu} t + \frac{1}{q\lambda + p\mu}\left(p\lambda + q\mu - \frac{\lambda\mu}{q\lambda + p\mu}\right) \times (1 - e^{-(q\lambda + p\mu)t}) \quad (1.1)$$

as long as $q\lambda + p\mu > 0$. Whitt (1982) gave a simple procedure for matching a mixture of exponentials to a given overdispersed distribution by using the first two moments.

Renewal functions for some other distributions have been tabulated for certain values of their parameters. See Baxter, Scheuer, McConalogue, and Blischke (1982) for a complete discussion. Life distributions whose renewal functions have been tabulated can also be used in the approximation procedure described here.

The main results in this article are for two renewal functions corresponding to two different life distributions. I obtain upper and lower bounds for the difference between the two renewal functions at each time point. The lower bound at a given time point is expressed in terms of a difference between the two underlying life distributions at the same time point. The upper bound is expressed in terms of the maximum difference between the two underlying life distributions over the time interval from 0 to the given time point, as well as reciprocals of the two survival functions. Moreover, I obtain an upper bound for the difference between the reciprocals of the two augmented renewal functions, again using the maximum difference of the two underlying life distributions over the same time interval. I show that the renewal functions are ordered in the same way that their life distributions (or their hazard rates) are. This leads me naturally to consider what other ordering properties may obtain between two renewal processes; I therefore discuss the conjecture that the map that takes a renewal process onto its associated renewal counting process reverses stochastic order (if two renewal processes are stochastically ordered, then their associated renewal counting processes are stochastically ordered in the reverse sense).

A renewal counting process is, of course, more comprehensively characterized by the distribution of the number of renewals than it is by the renewal function. It would, therefore, be useful to have bounds for the difference between two distributions of number of renewals, similar to those I give for renewal functions. To say that the development of such bounds is outside the scope of the present article is in no way to minimize the importance of being able to compare the whole distributions of number of renewals instead of just the renewal functions (i.e., their expected values).

The remainder of this article is organized as follows. Section 2 contains the statements of the results for the renewal function. Section 3 contains three examples illustrating the application of the results in Section 2. All mathematical details and proofs are in the Appendix.

2. STATEMENT OF RESULTS

2.1 Definitions and Preliminaries

Let F and G be life distributions—that is, cumulative distribution functions, continuous from the right, with $F(0^-) = G(0^-) = 0$. Denote the respective renewal functions (see Sec. 1) by M_F and M_G. As notation for the Stieltjes convolution, let $F * G(t)$ stand for $\int_{0^-}^{t} F(t-u)\, dG(u)$. For successive convolutions, define $F_1 = F$ and for $n > 1$, $F_{n+1} = F_n * F$. F_0 represents the identity for Stieltjes convolution—namely, the unit step function with step at 0. Then the renewal function M_F for F is given by

$$M_F(t) = \sum_{n=1}^{\infty} F_n(t) \tag{2.1}$$

(Karlin and Taylor 1975, sec. 5.1). The augmented renewal function includes a renewal at 0 and is given by $M_F^0 = F_0 + M_F$.

2.2 Upper Bounds

The goal is to express the difference between two renewal functions, which may be impossible or inconvenient to compute, in terms of the difference between their underlying life distributions, which should be easier to compute. In practice, we will have a renewal process with interrenewal time distribution F whose renewal function M_F is not known in closed form or tabulated. We then choose a distribution G that is uniformly close to F over the time interval of interest and whose renewal function M_G can be computed or is tabulated. We use $M_G(t)$ as an approximation to the unknown $M_F(t)$. Expressions like (2.2), (2.3), and (2.5) are then used to develop both a bound on the approximation error and upper and lower bounds for the unknown renewal-function value. This procedure is illustrated in (3.5) of Example 2 (Sec. 3).

The first upper bound expresses the difference between the two renewal functions, even if both are unknown, in terms of quantities that can be computed as long as F and G are known.

Proposition 1. For every $T > 0$ satisfying $F(T) < 1$, $G(T) < 1$,

$$|M_F(t) - M_G(t)| \le (1 - F(t))^{-1}(1 - G(t))^{-1}\|F - G\|_{\infty,T}, \quad 0 \le t \le T, \tag{2.2}$$

where $\|F - G\|_{\infty,T} = \sup_{0\le t\le T} |F(t) - G(t)|$.

In practice, if one had a value s at which this approximation was desired, then one would use (2.2) with $t = T = s$, because $\|F - G\|_{\infty,T}$ is nondecreasing in T. In this first result, the difference between the life distributions is the supremum of their absolute difference over the time interval of interest. Both the supremum and the reciprocal survival-function terms control how small the right side of (2.2) can be so that if T is too large, the bound may be less effective. If more information about F and G is available, the bound can be improved to involve only the absolute difference between the two life distributions at each point. This is the content of the next result.

Corollary 1. If $|F(t) - G(t)|$ is nondecreasing on an interval $[0, T]$ with $F(T) < 1$, $G(T) < 1$, then

$$|M_F(t) - M_G(t)| \le (1 - F(t))^{-1}(1 - G(t))^{-1}|F(t) - G(t)|, \quad 0 \le t \le T. \quad (2.3)$$

To obtain some increased precision for small values of t, we have traded away the generality of F and G that held in (2.2). Statement (2.3) holds only as long as the absolute difference between F and G does not decrease. In practice, this may be easiest to show when F and G are two members of the same parametric family of distributions (see Ex. 1, Sec. 3, for an illustration).

The final upper bound given is for the difference of the reciprocals of the augmented renewal functions for F and G. This result is of more theoretical than practical interest, because it has to be cast in the form (2.2) or (2.3) before it is good for obtaining an estimate of the difference between the renewal functions (without the reciprocals). Its proof (see the Appendix) shows, however, that this is the result from which all of the others derive.

Theorem 1. For every $T > 0$,

$$|M_F^0(t)^{-1} - M_G^0(t)^{-1}| \le \|F - G\|_{\infty, T}, \quad 0 \le t \le T. \quad (2.4)$$

2.3 Lower Bound

I offer only one simple lower bound for the difference between the two renewal functions. There is some more discussion in the Appendix about how this lower bound can be modified when more information is available.

Proposition 2. Suppose that there is a T for which $G(t) \le F(t)$ for all $t \in [0, T]$. Then

$$M_F(t) - M_G(t) \ge F(t) - G(t), \qquad 0 \le t \le T. \quad (2.5)$$

2.4 Ordering

The lower bound (2.5) leads to two order results, the second of which has become somewhat of a folklore result of reliability theory; it is that one can obtain a conservative approximation to the number of failures of a renewable system by applying the same model to a distribution function having an everywhere larger hazard rate. The formulation given here provides a simple analytic proof.

Theorem 2. If $F(t) \ge G(t)$ for all $t \in [0, T]$, then $M_F(t) \ge M_G(t)$ $(0 \le t \le T)$.

Corollary 2. Let h_F and h_G denote the hazard-rate functions for F and G, respectively. Suppose that $h_F(t) \ge h_G(t)$ for all $t \in [0, T]$. Then $M_F(t) \ge M_G(t)$ $(0 \le t \le T)$.

2.5 Review of Large-Time Asymptotic Results

Large-time limit theorems abound in renewal theory. Indeed, the subject can be said to be extensively concerned with what happens to systems modeled by renewal processes after initial transients have died out. It is the "early-time" transient behavior that is important for reliability modeling of highly reliable systems, however, because for these systems steady state may not be reached until long after the system's service life is over. To give a more complete picture of how to compare two renewal processes, I will review some large-time asymptotic results that complement the bounds given in Sections 2.2 and 2.3.

Let μ_F and μ_G denote the first moments of F and G, respectively, and let σ_F^2 and σ_G^2 denote their second central moments. Assume that these are all finite and that F and G are nonlattice or nonarithmetic [a distribution is arithmetic, or lattice, if it is concentrated only on multiples of a single positive number, called its span (see Feller 1971, p. 360)]. For large values of t, the elementary renewal theorem allows us to assert that $M_F(t) - M_G(t) \sim (\mu_F^{-1} - \mu_G^{-1})t$, as long as $\mu_F \ne \mu_G$. The renewal theorem also gives the second term in the asymptotic expansion

$$\lim_{t\to\infty} [M_F(t) - M_G(t) - t(\mu_F^{-1} - \mu_G^{-1})] = (\sigma_F^2 - \mu_F^2)/2\mu_F^2 - (\sigma_G^2 - \mu_G^2)/2\mu_G^2 \quad (2.6)$$

(Karlin and Taylor 1975, sec. 5.6b). This enables us to see what is happening when $\mu_F = \mu_G = \mu$. In that case, we have $M_F(t) - M_G(t)$ converging to $(\sigma_F^2 - \sigma_G^2)/2\mu^2$. We can also see that a necessary and sufficient condition for $M_F(t) - M_G(t)$ to converge to 0 is that F and G have equal means and equal variances.

For lattice distributions, the conclusions of the elementary renewal theorem only hold on the lattice points. Therefore, to be precise, these distributions have to be singled out for annoying, but unenlightening, special treatment. If F is lattice with span α and G is lattice with span β, then the left side of (2.6) has to be replaced by $\lim_{n\to\infty} [M_F(n\alpha) - M_G(n\beta) - n(\alpha\mu_F^{-1} - \beta\mu_G^{-1})]$. If only one of F or G is lattice with span α and the other is not lattice, then the left side of (2.6) has to be replaced by $\lim_{n\to\infty} [M_F(n\alpha) - M_G(n\alpha) - n\alpha(\mu_F^{-1} - \mu_G^{-1})]$. These technicalities cause additional bookkeeping in these special cases, but because the renewal function is supported on the same lattice that the distribution is, this has to be taken into account in the conclusions one would reach about the asymptotic behavior of the corresponding renewal processes.

In particular, if you choose a life distribution with a known renewal function for approximating another with unknown renewal function by matching the distributions' first two moments, the large-time behavior of the approximation becomes $M_F(t) - M_G(t) = o(1)$ as $t \to \infty$. If the means are equal, but the variances are not (this is the situation that prevails in Ex. 2 of Sec. 4), we get $M_F(t) - M_G(t) = O(1)$. If the means are not equal, the two renewal functions grow apart from each other like a constant times t. In any case, these are better than (2.2) would give for large t, reinforcing again the idea that the results of this section are most useful for small t.

2.6 Stochastic-Order Relations Between Renewal-Counting Processes

Finally, I discuss some results about stochastic order (Kamae, Krengel, and O'Brien 1977). This section does not help with the computation of bounds or approximations, but it provides more qualitative insight into the comparison of two renewal processes and their associated renewal-counting processes.

If X has distribution F and Y has distribution G, then we say that X is stochastically greater than Y ($X \geq_{st} Y$) if $\int a\, dF \leq \int a\, dG$ for every bounded increasing real-valued function a. If X and Y are processes, then $X \geq_{st} Y$ if $\int a\, dF \leq \int a\, dG$ for all the finite-dimensional distributions F of X and G of Y. Let $X = (X_1, X_2, \ldots)$ and $Y = (Y_1, Y_2, \ldots)$ be renewal processes having interrenewal time distributions F and G, respectively, and let $\{N_X(t) : t \geq 0\}$ and $\{N_Y(t) : t \geq 0\}$ be their respective renewal-counting processes. If $F(t) \geq G(t)$ for all $t \geq 0$, then $X_i \leq_{st} Y_i$ for every i, and, by independence, $X \leq_{st} Y$ as processes. Further, $\Pr\{N_X(t) \geq k\} = F_k(t) \geq G_k(t) = \Pr\{N_Y(t) \geq k\}$ for every t (see the beginning of the Appendix), so $N_X(t) \geq_{st} N_Y(t)$ as random variables for every t. In intuitive terms, if the interevent intervals of the X process tend to be shorter than those of the Y process, then the number of renewals in the X process will tend to be greater than the number of renewals in the Y process. Is it true, then, that the map that takes a renewal process onto its associated renewal-counting process always reverses stochastic order (of processes)? If X and Y have exponential interrenewal time distributions, then the answer is yes for the homogeneous Poisson processes $\{N_X(t)\}$ and $\{N_Y(t)\}$. This is false in general, however. Using the dependence of $N_X(t_1)$ and $N_X(t_2) - N_X(t_1)$, examples can be constructed that show that the two-dimensional distributions of N_X and N_Y need not be ordered even when X and Y are stochastically ordered.

3. EXAMPLES

This section contains three examples illustrating the applicability of the results of Section 2. The first is a simple one that illustrates the ideas without getting bogged down in too many computational details. The second is a more realistic example from a reliability model, although the answer can still be obtained in closed form for this example. The third example is similar to the second, but it involves lognormal distributions for which no closed-form solutions are available.

Example 1. Let $F(t) = 1 - e^{-\lambda t}$ and $G(t) = 1 - e^{-\mu t}$, and suppose that $\lambda > \mu$. Then $\sup_{0\leq t<\infty} |F(t) - G(t)|$ is attained at $T = (\ln \lambda - \ln \mu)/(\lambda - \mu)$. Because of this, (2.3) holds for $0 \leq t \leq T$, and we thereby obtain $(\lambda - \mu)t \leq e^{(\lambda+\mu)t}(e^{-\mu t} - e^{-\lambda t}) = e^{\lambda t} - e^{\mu t}$ for $0 \leq t \leq T$. For the lower bound, we observe that $\lambda > \mu$ implies that $F(t) \geq G(t)$ for all t, so Proposition 4 yields $(\lambda - \mu)t \geq e^{-\mu t} - e^{-\lambda t}$. Combining these, we obtain

$$e^{-\mu t} - e^{-\lambda t} \leq (\lambda - \mu)t \leq e^{\lambda t} - e^{\mu t}, \qquad 0 \leq t \leq T. \tag{3.1}$$

In fact, (3.1) holds over all of $[0, \infty)$; this follows from the intermediate-value theorem. The example suggests, again, that (2.3) and (2.5) are not likely to be very good for large t. As observed previously, however, this is not the region of major interest for these bounds.

The point of this example is not that we are learning something new from it about renewal functions for exponential distributions but rather to illustrate all of the approximation results obtained previously in a case in which all of the relevant quantities can be computed easily. In practical applications, of course, one would compute the renewal function for only one of $F(t)$ or $G(t)$. One would then have three known values [$F(t)$, $G(t)$, and $M_G(t)$, say] in (2.2), (2.3), or (2.5) with which to gain information about the fourth, unknown, value [in this case, $M_F(t)$]. This is illustrated in Examples 2 and 3.

Example 2. In a recent undersea fiber optics cable communications system design, each repeater contained equipment to enable two directions of transmission. This equipment consisted of three fibers and associated regenerators that were otherwise unrelated, two of which were active and the third of which was operated in hot standby to improve overall reliability. Each direction of transmission in each repeater, then, was modeled as a two-out-of-three hot standby ensemble of identical, stochastically independent units. We will begin this example with the simple assumption that each regenerator has life distribution $1 - e^{-\lambda t}$; we will replace this with a more realistic assumption in Example 3. In this design (different from the one described in Sec. 1) the least replaceable unit is the ensemble; that is, all three units are turned on at time 0, and at the time of the second unit failure, the entire ensemble or system of three units is instantaneously replaced with a new one. In reality, of course, the time it takes to do this replacement is nonzero, but it is short enough compared with the expected ensemble life that the pure renewal process model gives good results. Operation continues in this way indefinitely. The reliability model for this system is a renewal process, with the time between renewals governed by the system life distribution. If the unit lifetimes are X_1, X_2, X_3, then the system life is $X_{(2)}$, the second-order statistic of the three X's. Then, ignoring possible switching device unreliability, the system life distribution is

$$F(t) = \Pr\{X_{(2)} \le t\} = 3(1 - e^{-2\lambda t}) - 2(1 - e^{-3\lambda t}). \tag{3.2}$$

From this, the expected system life is $5/6\lambda$, and the variance of the system life is $13/36\lambda^2$. This distribution is underdispersed, so we have a chance for success if we choose an appropriate Erlang distribution for comparison. The Erlang distribution of order k, whose distribution function is

$$1 - \left(1 + \mu t + \frac{(\mu t)^2}{2!} + \cdots + \frac{(\mu t)^{k-1}}{(k-1)!}\right) e^{-\mu t},$$

has mean k/μ and variance k/μ^2. The Erlang distribution whose first two moments most closely match those of F is that of order two—namely, $G(t) = 1 - (1 + \mu t)e^{-\mu t}$, with $\mu = 12\lambda/5$. Its mean matches that of F exactly, and its variance is $25/72\lambda^2$, which differs from the variance of F by less than 4% relative error. From Parzen [1962, chap. 5, eq. (2.28)], I obtain the renewal function for G as

$$M_G(t) = \frac{6\lambda}{5} t - \frac{1}{4}(1 - e^{-24\lambda t/5}). \tag{3.3}$$

Suppose now that the system design life, or service life, is $t_D = 25$ years (219,150 hours), that λ is equal to 2.3 failures per million hours (2,300 failures in 10^9 hours), and that I want to estimate the expected number of system failures over the system design life. This latter quantity is $M_F(t_D)$. $F(t) - G(t)$ is positive and increasing over the interval $[0, t_D]$ (even though it is only elementary calculus, this is where the greatest amount of work needs to be done in establishing the bounds). Thus by Proposition 2 and Corollary 1, we may write

$$\begin{aligned} F(t) - G(t) &\le M_F(t) - M_G(t) \\ &\le (1 - F(t))^{-1}(1 - G(t))^{-1}(F(t) - G(t)) \end{aligned} \tag{3.4}$$

for all t between 0 and t_D. In particular, (3.4) holds for $t = t_D$. Evaluating $F(t_D) = .34613$ from (3.2), $G(t_D) = .34088$, and $M_G(t_D) = .37710$ from (3.3), we obtain from (3.4)

$$.00525 \le M_F(t_D) - .37710 \le .01218, \tag{3.5}$$

or $.38235 \le M_F(t_D) \le .38928$. The spread in this bound is less than 2%, and it should, therefore, be very useful in approximating $M_F(t_D)$ for any practical purpose.

The large time limit of $M_F(t)/t$ is 2.760×10^{-6}. If we were to use this large time limit to approximate $M_F(t_D)$, we would obtain $M_F(t_D) \approx .60485$, which is conservative but is 57% too big. In this example, the components of the system are highly reliable, and the system's failure process does not reach a steady-state condition until long after service life is over.

In this case, one can get the renewal function for F from (1.1); it is

$$M_F(t) = \frac{6\lambda}{5} t - \frac{6}{25}(1 - e^{-5\lambda t}).$$

From this, we obtain $M_F(t_D) = .38416$, which is squarely between the two bounds obtained previously.

Example 3. Consider again the two-out-of-three hot standby ensemble of regenerators described in Example 2, except that now take the unit life distribution $L(t)$ to be lognormal with mean $1/\lambda$ and variance $1/\lambda^2$ [this gives a median of $1/(\lambda\sqrt{2})$ and a spread factor of $(\ln 2)^{1/2}$]. The lognormal distribution often has been used to describe the reliability of various semiconductor devices, both discrete and integrated (Peck and Zierdt 1974). Since the component that dominates the reliability of the regenerator is the semiconductor laser and its life has been modeled with the lognormal distribution, we will take the regenerator life distribution to be lognormal even though the true regenerator life distribution may deviate slightly from lognormal because of the effects of its other components. Again the least replaceable unit is the whole ensemble. Ignoring possible switching device unreliability, the system life distribution is $F(t) = 3L(t)^2 - 2L(t)^3$. We want to find the value of the renewal function M_F for F at the end of the service life t_D, again taken as 219,150 hours. Using

λ^{-1} = 434,782.61 hours, the same mean as in Example 2, we obtain $F(t_D)$ = .23217. The mean and variance of F are unknown in this case, however, so a certain amount of trial and error is needed. This is facilitated by use of the S statistical programming language to evaluate the lognormal distributions and to draw appropriate graphs.

We will again use an Erlang distribution of order two for the comparison distribution G. For values of μ that make $G(t_D)$ reasonably close to $F(t_D)$, F and G cross somewhere between 0 and t_D, so we cannot use (2.3) or (2.5). We will use (2.2) instead. Then the strategy becomes to choose μ so that $\|F - G\|_{\infty,t_D}$ is as small as possible. It is reasonable here to use $\mu = 3.6 \times 10^{-6}$, yielding $G(t_D)$ = .18724 and $\|F - G\|_{\infty,t_D}$ = .07851. From (2.2), we then obtain $|M_F(t_D) - .19607| \le .07851$, from which it follows that $.11756 \le M_F(t_D) \le .27458$. For the lower bound, clearly we can do much better using the simple inequality $M_F(t_D) \ge F(t_D)$, so finally we obtain $.23217 \le M_F(t_D) \le .27458$. The spread in these bounds is about 18%, which is not as good as we got in Example 2. Here, however, we have to take into account the qualitative differences between F, an algebraic combination of lognormal distributions, and the comparison distribution, which we restricted to be an Erlang of order two for comparison purposes with Example 2. Perhaps a shifted exponential distribution would work better as the comparison distribution, because F does not get bigger than .001 until nearly 55,000 hours have passed. Intuitively, the shifted exponential also has a better chance of staying below F over $[0, t_D]$. If so, (2.5) could be used to obtain a lower bound.

4. SUMMARY AND CONCLUSIONS

We have obtained upper and lower bounds for the difference between two renewal functions corresponding to two different life distributions. These results are used to examine various order relationships that hold for the renewal functions based on corresponding order relationships that hold for their underlying life distributions. These also lead to a simple proof of the intuitively satisfying fact that if the hazard rate of F is never exceeded by that of G, then the renewal function for F is never exceeded by that for G. The results are also used to provide an approximation procedure for an unknown or difficult-to-compute renewal function in terms of a known or easier-to-compute renewal function. The procedure also gives bounds on the error of approximation.

ACKNOWLEDGMENT

I thank the editors and the referees for suggestions that helped improve this article.

APPENDIX: PROOFS OF THE RESULTS IN SECTION 2

I begin by stating some simple properties of Stieltjes convolution. First, it is commutative, associative, and distributive over addition. Second, $F * G(t) \le F(t)G(t)$. This follows by $F * G(t) = \int_{0^-}^{t} F(t - u)\, dG(u) \le F(t) \int_{0^-}^{t} dG(u) \le F(t)G(t)$. Third, if $G(t) \le F(t)$ for all $t \in [0, T]$, then $G_n(t) \le F_n(t)$ for all $t \in [0, T]$. To see this, note that $G_2(t) = G * G(t) \le F * G(t) = G * F(t) \le F * F(t) = F_2(t)$ and proceed by induction.

Since the proof of Proposition 2 follows almost immediately from this, I will take the opportunity to record it now, even though it is somewhat out of sequential order.

Proof of Proposition 2. We have, for all appropriate t,

$$M_F(t) - M_G(t) = F(t) - G(t) + \sum_{n=2}^{\infty} [F_n(t) - G_n(t)] \ge F(t) - G(t), \quad \text{(A.1)}$$

because the sum is nonnegative by the inequality stated previously.

The same argument shows that, under the same conditions,

$$M_F(t) - M_G(t) \ge \sum_{n=1}^{k} [F_n(t) - G_n(t)], \qquad k \ge 1,$$

although this is somewhat less practical than (A.1) as a simple computational tool when $k > 1$. Theorem 2 follows immediately from Proposition 2. Corollary 2 follows by noting that ordering of the hazard-rate functions entails ordering of the corresponding life distributions through the equality $1 - F(t) = \exp(-\int_0^t h_F(s)\, ds)$.

Theorem 1 is presented after Proposition 1 and Corollary 1 in the text because it is less suited for immediate computational use. I will give the proof of Theorem 1 first, however, because Proposition 1 and Corollary 1 follow quickly from it.

Proof of Theorem 1. Begin by showing that

$$|F_n(t) - G_n(t)| \le \|F - G\|_{\infty,T} \sum_{k=0}^{n-1} F_k * G_{n-1-k}(t). \quad \text{(A.2)}$$

Do this by induction. For $n = 1$, this follows from the definition of the norm. Assume the induction hypothesis for n. Then $F_{n+1}(t) - G_{n+1}(t) = (F -$

$G) * F_n(t) + (F_n - G_n) * G(t)$ by adding and subtracting the needed term. Here is the induction step:

$$\begin{aligned}|F_{n+1}(t) - G_{n+1}(t)| &\le \int_{0^-}^{t} |F(t-u) - G(t-u)|\,dF_n(u) \\ &\quad + \int_{0^-}^{t} |F_n(t-u) - G_n(t-u)|\,dG(u) \\ &\le \|F - G\|_{\infty,T} F_n(t) \\ &\quad + \int_{0^-}^{t} \|F - G\|_{\infty,T} \sum_{k=0}^{n-1} F_k * G_{n-1-k}(t-u)\,dG(u) \\ &= \|F - G\|_{\infty,T} \left(F_n(t) + \sum_{k=0}^{n-1} F_k * G_{n-k}(t) \right) \\ &= \|F - G\|_{\infty,T} \sum_{k=0}^{n} F_k * G_{n-k}(t).\end{aligned}$$

Now complete the demonstration of (2.4) as follows:

$$\begin{aligned}|M_F(t) - M_G(t)| &\le \sum_{n=1}^{\infty} |F_n(t) - G_n(t)| \\ &\le \|F - G\|_{\infty,T} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} F_k * G_{n-1-k}(t) \\ &= \|F - G\|_{\infty,T} \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} F_{k-1} * G_{n-k}(t) \\ &= \|F - G\|_{\infty,T} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} F_{k-1} * G_{n-1}(t) \\ &= M_F^0 * M_G^0(t) \|F - G\|_{\infty,T},\end{aligned}$$

from which (2.4) follows by observing that $M_F^0 * M_G^0(t) \le M_F^0(t) M_G^0(t)$ because M_F^0 and M_G^0 are nondecreasing and then collecting terms.

Proposition 1 follows immediately from the observation that

$$M_F^0(t) = \sum_{n=0}^{\infty} F_n(t) \le \sum_{n=0}^{\infty} F(t)^n = (1 - F(t))^{-1}$$

whenever $F(t) < 1$. Corollary 1 then follows by observing that since $|F(t) - G(t)|$ is nondecreasing on $[0, T]$, $\sup_{0 \le s \le t} |F(s) - G(s)| = |F(t) - G(t)|$ for every $t \in [0, T]$.

[Received September 1986. Revised August 1988.]

REFERENCES

Barlow, R. E., and Proschan, F. (1965), *Mathematical Theory of Reliability*, New York: John Wiley.

Baxter, L. A., Scheuer, E. M., McConalogue, D. J., and Blischke, W. R. (1982), "On the Tabulation of the Renewal Function," *Technometrics*, 24, 151–156.

Cleroux, R., and McConalogue, D. J. (1976), "A Numerical Algorithm for Recursively-Defined Convolution Integrals Involving Distribution Functions," *Management Science*, 22, 1138–1146.

Deligonul, Z. S., and Bilgen, S. (1984), "Solution of the Volterra Equation of Renewal Theory With the Galerkin Technique Using Cubic Splines," *Journal of Statistical Computation and Simulation*, 20, 37–45.

Feller, W. (1971), *An Introduction to Probability Theory and Its Applications* (Vol. 2, 2nd ed.), New York: John Wiley.

Kamae, T., Krengel, U., and O'Brien, G. L. (1977), "Stochastic Inequalities on Partially Ordered Spaces," *The Annals of Probability*, 5, 899–912.

Kao, E. P. C. (1988), "Computing the Phase-Type Renewal and Related Functions," *Technometrics*, 30, 87–93.

Karlin, S., and Taylor, H. M. (1975), *A First Course in Stochastic Processes* (2nd ed.), New York: Academic Press.

McConalogue, D. J. (1981), "Numerical Treatment of Convolution Integrals With Densities Having Singularities at the Origin," *Communications in Statistics, Part B—Simulation and Computation*, 10, 265–280.

Neuts, M. F. (1981), *Matrix-Geometric Solutions in Stochastic Models*, Baltimore: Johns Hopkins University Press.

Parzen, E. (1962), *Stochastic Processes*, San Francisco: Holden-Day.

Peck, D. S., and Zierdt, C. H. (1974), "The Reliability of Semiconductor Devices in the Bell Sytstem," *Proceedings of the Institute of Electrical and Electronic Engineers*, 62, 185–211.

Soland, R. M. (1969), "Availability of Renewal Functions for Gamma and Weibull Distributions With Increasing Hazard Rate," *Operations Research*, 17, 536–543.

Thompson, W. A. (1981), "On the Foundations of Reliability," *Technometrics*, 23, 1–13.

Tortorella, M. (1987), "Newton–Cotes Quadrature Rules for Stieltjes Integrals and Numerical Solution of Integral Equations of Renewal Type," unpublished memo, AT&T Bell Laboratories, Whippany, NJ.

Whitt, W. (1982), "Approximating a Point Process by a Renewal Process, I: Two Basic Methods," *Operations Research*, 30, 125–147.