Optimum Accelerated Life Tests With a Nonconstant Scale Parameter

Carol A. Meeter
Statistical Consulting
3M Center
St. Paul, MN 55144

William Q. Meeker, Jr.
Department of Statistics
Iowa State University
Ames, IA 50011

Previous work on planning accelerated life tests has assumed that the scale parameter \( \sigma \) for a location-scale distribution of log lifetime remains constant over all stress levels. This assumption is, however, inappropriate for many applications, including accelerated tests for metal fatigue and certain electronic components. This article extends the existing maximum likelihood theory for test planning to the nonconstant \( \sigma \) model and presents test plans for a large range of practical testing situations. The test plans are optimum in that they minimize the asymptotic variance of the maximum likelihood estimator of a specified quantile at the design stress. The development and discussion in the article, as well as the theory given in the Appendix, applies to accelerated-life-test models in which the log time-to-failure can be modeled as a location-scale distribution. The test setup assumes simultaneous testing of units with time censoring. We give particular numerical results for the Weibull failure time distribution.

KEY WORDS: Censored data; Experimental design; Life-data analysis; Life-test planning; Weibull distribution; Reliability.

1. INTRODUCTION

1.1 Accelerated Life Tests

Estimating the time-to-failure distribution of components of high-reliability products is difficult; few units will fail in a life test of practical length at normal use conditions. Life distributions can, however, be estimated much more quickly using accelerated life tests (ALT's).

Consider the reliability of a toaster, which if used twice each day has an expected lifetime of 20 years. Using the toaster 365 times each day would reduce the expected lifetime to about 40 days. For many products and materials, however, it is impossible to increase the usage rate. In such situations, it is sometimes possible to accelerate life by testing at stress levels that are higher than normal. Typical accelerating stresses include temperature, humidity, voltage, and pressure. Assuming a physically reasonable statistical model relating lifetime to level of stress, the life data from the accelerated test can be used to estimate a failure-time distribution under normal stress levels. Nelson (1990) provided a comprehensive review of the theory and practice of accelerated life testing.

Most of the previous statistical research on accelerated life testing has assumed that log life follows a location-scale distribution for which the location parameter \( \mu \) varies with stress and for which the scale parameter \( \sigma \) remains constant. The assumption that \( \sigma \) does not depend on stress is, however, inappropriate for many applications. The literature on metal fatigue, electronics reliability, and reliability physics contains many such applications. Nelson (1984) and Boyko and Gerlach (1989) analyzed data showing clearly that \( \sigma \) depends on stress. For other discussions of models and physical situations leading to data with nonconstant \( \sigma \), see Schreiber and Grabe (1981), Glaser (1984), Joyce, Liou, Nash, Bossard, and Hartman (1985), McPherson and Baglee (1985), Schwarz and Felton (1985), Schwarz (1987), Hiegeist, Spitzer, and Röhl (1989), Chan (1991), Li, Ting, and Kwong (1989), and Nelson (1990, p. 105).

In this article, we extend the previous work on planning ALT's with time (Type I) censoring to the case in which the scale parameter \( \sigma \) depends on stress.

1.2 Related Work

There is a considerable amount of literature available on planning ALT's with censored data. We will review some of the work most closely related to this article; see Nelson (1990, chap. 6) for a more complete review.

Chernoff (1962) developed "locally" optimum ALT's for the exponential distribution to estimate failure rate at a specified design stress. He assumed...
two forms for the failure rate, a quadratic function and an exponential function of stress. He gave results for both simultaneous testing of a sample of units until a prespecified time (allowing for time censoring) and for successive testing in which units are tested one at a time until they fail. Chernoff called his plans "locally optimum" because they depend on the true (unknown) parameter values. Most optimum designs associated with nonlinear estimation problems (including estimation with censored data) result in locally optimum designs. Hereafter we will omit the term "local" in our discussion of optimum plans.

Nelson and Kielpinski (1975, 1976) gave optimum plans and best traditional plans (traditional plans use equally spaced levels of stress and equal allocation of test units to the different levels of stress) for the median of a normal or lognormal distribution. Their model assumes that the normal distribution location parameter \( \mu \) (also the mean) is a linear function of stress and that the scale parameter \( \sigma \) (also the standard deviation) does not depend on stress. They also assume simultaneous testing of all test units and censoring at a prespecified time.

Nelson and Meeker (1978) gave similar optimum plans to estimate quantiles of Weibull and smallest extreme-value distributions. They assumed that the smallest extreme-value location parameter \( \mu \) (also the .632 quantile) is a linear function of stress and that the scale parameter \( \sigma \) does not depend on stress. They also assume simultaneous testing of all test units and censoring at a prespecified time.

Using assumptions similar to those that are used in these references, Meeker (1984) compared the statistically optimum test plans to more practical test plans that have three levels of stress. Meeker and Hahn (1985) provided extensive tables and practical guidelines for planning ALT's, presented in a format to be used by engineers. Their tables allow one to assess the effect that reducing the testing stress levels (thereby reducing the degree of extrapolation) will have on statistical precision. Jensen and Meeker (1990) provided a computer program that allows the user to develop and compare optimum and compromise ALT plans. The program also allows the user to modify or specify plans and to evaluate their properties.

1.3 Overview

In Section 2, we describe our model and assumptions. Section 3 outlines a convenient standardization and reparameterization of the model. In Section 4, we describe optimum and compromise plans for the nonconstant \( \sigma \) model, give tables of different ALT plans, and discuss the results. Section 5 describes a numerical example. In Section 6, we make some concluding remarks. The Appendix contains maximum likelihood theory for evaluation and optimization of test plans.

2. MODEL ASSUMPTIONS AND TEST STRESSES

The model assumptions used in this article are:

1. Lifetimes of test units are independent, and at any stress level, product life has a Weibull distribution. Thus the log failure time \( Y \) has a smallest extreme-value distribution with a cdf

\[
Pr(Y \leq y) = F(y; \mu, \sigma) = \Phi_{\text{sev}} \left( \frac{y - \mu}{\sigma} \right), \quad -\infty < y < \infty,
\]

where \( \sigma > 0 \) is the scale parameter of log life, \( -\infty < \mu < \infty \) is the location parameter of log life, and \( \Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)] \) is the standardized smallest extreme-value distribution. For the lognormal model, one can simply use the standard normal cdf \( \Phi_{\text{std}}(z) \) in place of \( \Phi_{\text{sev}}(z) \). We use \( \Phi(z) \) generically.

2. The location parameter, \( \mu \), is a linear function of the stress level \( \mu(x) = \gamma_0 + \gamma_1 x \), where \( x \) is some (possibly transformed) measure of stress (e.g., for the Arrhenius model, \( x \) is chosen to be proportional to the reciprocal of absolute temperature). We refer to \( x \) simply as stress. The commonly used Arrhenius, Inverse Power Rule, and Eyring ALT models (see Nelson 1990, chap. 2) are special cases of this relationship.

3. The scale parameter is a log-linear function of stress. That is, \( \log(\sigma(x)) = \gamma_0^{\sigma} + \gamma_1^{\sigma} x \).

The standard constant \( \sigma \) accelerated-failure-time model can be justified on the basis of a physical model in which activation energy is constant and unit-to-unit variability is caused by variability in the pre-exponential factor of the Arrhenius relationship (e.g., Klinger 1992). Joyce et al. (1985) suggested a log-linear relationship for \( \sigma \) in an Arrhenius-type temperature-acceleration model for laser life. Their model is based on experimental evidence that activation energy is random from unit to unit. For temperature acceleration of electromigration in integrated circuits, Schwarz (1987) suggested an alternative model in which both the activation energy and the pre-exponential factor in the Arrhenius relationship are random, but with a correlation of \(-1\). Schwarz's model can also be transformed into a log-linear model for \( \sigma \). These two physical models were described, compared, and generalized by Chan (1991). Nelson (1984) used a log-linear relationship for \( \sigma \) in the analysis of fatigue data.

In addition to the model assumptions, we have the following assumptions about how the test is conducted.

1. The ALT involves simultaneous testing of all units until a prespecified log censoring time, \( \eta \). Thus
the censoring time is fixed and the number of failures is random (time censoring).

2. The failure times of the test units (including those that we do not observe due to censoring) are statistically independent.

The unknown parameters $\gamma_0$, $\gamma_1$, $\gamma_2^{[\nu]}$, and $\gamma_3^{[\nu]}$ describe properties of the product and must be estimated from the ALT data. We assume that interest centers on $Y_P(x) = \mu(x) + u_0\sigma(x)$, the $P$ quantile of the distribution of log life at stress $x = x_D$, the design stress. For the Weibull-distribution model, $u_P = \Phi^{-1}(P) = \log(-\log(1 - P))$ is the cdf of the standard smallest extreme-value distribution. Because few, if any, failures are expected at $x_D$ before the end of the test, it is inefficient to test at or near there.

The highest test stress level, $x_H$, should be fixed as high as possible, subject to the constraint that the physical model remains adequate. Although increasing $x_H$ will improve estimation precision, bias can result if the physical model no longer holds at the higher level. We let $x_D$ denote the lowest test stress level. When there are subexperiments at three levels of stress, we let $x_M$ denote the intermediate test level.

3. MODEL STANDARDIZATION AND REPARAMETERIZATION

In this section we present a standardization and reparameterization that will allow us to present our general results more concisely.

3.1 Standardized Stress

We define standardized stress as

$$\xi = \xi(x) = \frac{x - x_D}{x_H - x_D}$$

so that $\xi_D = 0$ and $\xi_H = 1$. In terms of standardized stress, our model is $\mu(\xi) = \beta_0 + \beta_1\xi$, $\log[\sigma(\xi)] = \beta_0^{[\nu]} + \beta_1^{[\nu]}\xi$.

3.2 Reparameterization

The following parameterization allows us to describe the accelerated testing situation with only three parameters and $P$, the quantile of interest. Let $\sigma(\xi) = \sigma_D\theta^\xi$, where $\theta = \sigma_H/\sigma_D$ is the ratio of the scale parameters at the high and design stresses. Any two of the following three parameters describe the relationship between $\sigma$ and $\xi$: $\sigma_D = \exp(\beta_0^{[\nu]})$, $\sigma_H = \exp(\beta_1^{[\nu]})$, $\theta = \exp(\beta_0^{[\nu]})$.

By standardizing, the test plans and their scaled properties can be characterized for the full range of testing situations by using the following standardized parameters:

1. The standardized log censoring times, $a = (\eta - \beta_0)/\sigma_D = (\eta - \mu_D)/\sigma_D$
2. The standardized slope, $b = \beta_1/\sigma_D = (\mu_H - \mu_D)/\sigma_D$
3. The ratio of scale parameters, $\theta = \sigma_H/\sigma_D$

In particular, as shown in the Appendix, the scaled asymptotic variance $(n/\sigma_D^2)\text{var}(\hat{Y}(\xi_D))$ depends on $a$, $b$, $\theta$, $P$, and the proposed test plan. Here $n$ is the total sample size for the ALT. Alternatively, the testing situations can be characterized by $p_D = \Phi(a)$, $p_H = \Phi((a - b)/\theta)$, and $\theta$. Then $p_M$ are the probabilities that a unit will fail by the end of the test at $\xi_D$ and $\xi_H$, respectively.

Following Meeker and Nelson (1975) and Nelson and Meeker (1978), we show how to develop optimum test plans that minimize the asymptotic variance of $\hat{Y}(\xi_D)$, the $P$ quantile of the life distribution at $\xi_D$. The test plan is specified by levels of stress and the proportionate allocations of units to these levels. In the most general case, we will have tests at stress levels $\xi_L$, $\xi_M$, and $\xi_H$ (where $\xi_L < \xi_M < \xi_H = 1$) with corresponding allocations $\pi_L$, $\pi_M$, and $\pi_H = 1 - (\pi_L + \pi_M)$.

The true values of $p_D$, $p_H$, and $\theta$ are, in general, unknown, but initial guess values of the parameters must be used to evaluate or optimize a test plan. These initial guesses are usually chosen on the basis of previous experience, preliminary experiments, or engineering judgment. The optimum plans were obtained using a numerical algorithm that is similar to that described by Meeker and Nelson (1975).

4. OPTIMUM AND COMPROMISE TEST PLANS AND COMPARISONS

Previous work in planning ALT's has assumed that $\sigma$ does not depend on stress. In our setup, this is equivalent to assuming that $\theta$ is equal to 1 and that $\theta$ will not be estimated from the ALT data. In developing test plans, we generalize from this model in two ways. First we consider models in which $\theta \neq 1$ and where $\theta$ will not be estimated from the data. Then we consider models in which $\theta$ is unknown and will be estimated from the data.

4.1 Optimum Plans When $\theta$ Will Not Be Estimated

When $\theta$ is assumed to be known and will not be estimated from the ALT data, the estimation and optimization problems are similar to those of the classical constant $\sigma$ case (e.g., Nelson 1990, chap. 6). Such an assumption could be justified, for example, by having available information on the unit-to-unit variability in activation energy for a particular device.

Estimation of the simple linear relationship for $\mu$ requires subexperiments at two (or more) different
levels of stress. For some testing situations, when finding an optimum test plan, $\pi_L \to 1$ and $\xi_L \to \xi_D$, approaching a degenerate test plan at $\xi_D$. Such a degenerate plan cannot estimate the regression relationship. Figure 1 shows a contour plot for the asymptotic variance surface for the testing situation $p_D = .01, p_H = .25$, and $\theta = .8$, for $P = .01$. The contours on this plot represent orders of magnitude increase in variance. For example, a contour value of 1 (2) indicates that the variance is 10 (100) times larger than the variance at the minimum of the variance function. For the case in Figure 1, it is best to test all units at the design stress. For such an experiment, the regression model parameters are not estimable. The asymptotic variance of $\hat{Y}_p(\xi_D)$ afforded by the single subexperiment at $\xi_D$ is, however, smaller than that provided by any plan that can estimate the regression model parameters. A degenerate plan will be optimum when there is little acceleration ($p_H$ not large enough relative to $p_D$) or when $p_D$ is large (e.g., $p_D > .10$).

In other situations, an ALT using two or more levels of stress can provide a smaller value of var($\hat{Y}_p(\xi_D)$). In such situations, when $\theta$ will not be estimated, it is optimum to use just two levels of stress and the highest one should be chosen to be as high as possible (if it is chosen to be too high, however, irrelevant failure modes may occur or the model might otherwise break down). Nelson and Kielpinski (1976) gave a heuristic argument for this result, based on their numerical results for the case when $\theta = 1$ is assumed to be known. Using similar numerical evaluations, we have determined that their argument extends to other values of $\theta$, when $\theta$ is assumed to be known and thus not to be estimated. Thus $\xi_H = 1$ is fixed and the optimization is with respect to $\xi_L$, the lowest test level of stress, and $\pi_L$, the proportion of units allocated to $\xi_L$.

Figure 1 shows a contour plot for the testing situation $p_D = .01, p_H = .99$, and $\theta = .8$, for $P = .001$. In this case, an ALT has a smaller asymptotic variance than a test with all units at the design stress.

4.2 Optimum Plans When $\theta$ Will Be Estimated

When $\theta$ will be estimated from the ALT data, there are again combinations of $p_D, p_H, \theta,$ and $P$ for which it will be optimum to use a degenerate plan that tests all units at the design stress.

With subexperiments at two different levels of stress, one can estimate the parameters in the linear relationships for both $\mu$ and $\log(\sigma)$, and for some combinations of $p_D, p_H, \theta,$ and $P$, using two subexperiments will be optimum. Through numerical experiments, however, we have found other combinations for which it is optimum to use subexperiments at three different levels of stress. Our tables of optimum plans show examples of three-stress optimum plans converging to two-stress optimum plans ($\pi_M \to 0$) and two-stress optimum plans converging to the degenerate plan at the design stress ($\pi_L \to 1$ and $\xi_L \to 0$). In either case, it will again be optimum to choose $x_H$ to be as high as possible.

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**Figure 1.** Variance Ratios for Nonconstant $\sigma$ With $P = .01, p_D = .01, p_H = .25$, and Known $\theta = .8$.

**Figure 2.** Variance Ratios for Nonconstant $\sigma$ With $P = .001, p_D = .01, p_H = .99$, and Known $\theta = .8$.  

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4.3 Tabulations

We provide tables for statistically optimum plans for the case in which \( \theta \) is assumed known and for the case in which \( \theta \) is to be estimated. For purposes of comparison, we also computed and evaluated three-level compromise plans similar to those suggested and extensively tabled by Meeker and Hahn (1985) for the constant \( \sigma (\theta = 1 \text{ assumed known}) \) case. Their goal in presenting the compromise plans was to overcome the practical deficiencies of true optimum plans and to maintain reasonable statistical efficiency, relative to the optimum plans. The compromise plans tend to be more robust to departures from assumptions used to determine the test plans. The compromise plans also provide information to detect departures from the assumed linear model. These compromise plans, called “optimized 4:2:1 allocation” plans, choose \( \xi_L \) to minimize the asymptotic variance of \( \hat{Y} \) under the restriction that test units are allocated to \( \xi_L, \xi_M, \) and \( \xi_H \) in ratios of 4:2:1, respectively, where \( \xi_M = (\xi_L + \xi_H)/2 \). As described by Meeker and Hahn (1985), these allocation ratios were chosen on the basis of extensive numerical evaluations of different test plans and different evaluation criteria and were found to perform well over the practical range of testing situations.

Although we did evaluations for a much wider range of inputs, Tables 1-4 present information on both the statistically optimum plan and the optimized 4:2:1 allocation plan for all combinations of

- \( P = .0001, .0010, \) and \( .0100 \)
- \( p_D = .0001, .0010, .0015 \)

subject to \( P \approx p_D \), for \( \theta \) assumed known and \( \theta \) assumed unknown. In most reliability applications, interest centers on the lower tail of the failure-time distribution. Thus, in our optimizations and evaluations, we use \( P \) ranging between .0001 and .01. We chose values of \( \theta \approx .80 \) and 1.00, based on applications (e.g., Boyko and Gerlach 1989; Nelson 1984) that we have seen, always giving \( \sigma_{\theta D} > \sigma_{\theta} \) and \( \theta \) in this range. In Section 5 we examine the effect of \( \theta \) over a wider range of values for a particular example.

For the statistically optimum plan, the tables show

- \( \xi_L \) — the optimum standardized low stress level
- \( \xi_M \) — the optimum standardized medium stress level (when \( \theta \) is assumed known and in some cases when \( \theta \) is to be estimated, \( \pi_M = 0 \) is optimum and thus there is no middle stress condition)
- \( \pi_L, \pi_M \) — the proportion of units allocated to the low and middle stress levels (the remaining proportion is allocated to the high stress level, \( \xi_H \))
- \( p_L, p_M \) — the probability of failure by log time \( \eta \) at the low and middle stress levels
- \( E_L, E_M, E_H \) — the expected number of failures by log time \( \eta \) at each stress level, per 1,000 units tested in the entire program
- \( V(\xi) \) — asymptotic variance of the estimator of the \( P \) quantile of the time-to-failure distribution at the design stress, multiplied by \( n/\hat{\sigma}_D^2 \)

Table 1. Test Plans Assuming a Weibull Distribution With \( \theta = 1 \) Known

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Table 1. Test Plans Assuming a Weibull Distribution With \( \theta = 1 \) Known

ACCELERATED LIFE TESTS WITH NONCONSTANT SCALE

[Technometrics, February 1994, Vol. 36, No. 1]
Table 2. Test Plans Assuming a Weibull Distribution With $\eta = .8$ Known

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and Hahn's (1985) test plans can be compared directly to the plans for the nonconstant $\sigma$ model given in Table 1. When $\theta$ is assumed to be known and equal to 1 (i.e., $\sigma$ is assumed to be constant), our results agree with those of Meeker and Hahn (1985).

4.4 Discussion

During our evaluations of the accelerated test plans, we computed numerous tables and graphs of test plans and properties for the Weibull and lognormal distributions. Because the results for the two different distributions were qualitatively similar, we only present detailed results for the Weibull distribution. The following discussion contains general observations that were gleaned from the tables:

1. General observations

- For the statistically optimum plan, the optimum $\pi_1$ is usually quite large, ranging from .6 to 1.0. This is as expected because the failure probability is smaller at $\xi_L$ and because $\xi_L$ is closer to $\xi_D$, where the $Y_P$ is to be estimated.
- For any fixed value of $p_D$, var($Y_P$) decreases as $p_H$ increases. That is, using more acceleration decreases the asymptotic variance of the estimator.
- In a few situations the optimum plan degenerates to a test with all units at $\xi_N$. This typically occurs when $p_D$ is large and the values of $p_D$ and $p_H$ are too similar to provide much acceleration. This only occurs outside of the range of our tables for $\theta$ assumed known but can be observed in the tables for $\theta$ assumed unknown.

2. Observations for the case when $\theta$ is assumed known

- For fixed $p_D$ and $p_H$, $\xi_L$ increases as $\theta$ decreases, regardless of the test plan. That is, the relative size of the scale parameter at the lower stress level increases, more information about the lower tail of the distribution is available from testing at higher levels of stress.
- For small $P (< .01)$, the value of $p_L$ for the optimum plan does not depend strongly on $\theta$. Similarly, when $P = p_D$, the value of $p_L$ corresponding to the optimum plan depends strongly on $p_H$ but hardly at all on the value of $P = p_D$. High-precision computations show, however, that the relationships are not exact.

3. Observations for the case when $\theta$ is assumed unknown

- In many cases, the statistically optimum plan reduces to a two-stress plan (i.e., the optimum value of $\pi_1 \to 0$). Three subexperiments are optimum when $P$ is very small and $p_D$ and $p_H$ are such that significant acceleration is possible (i.e., with small $p_D$ and large $p_H$).
- For a given $p_D$ and $P$ and small values of $p_H$, $\xi_L$ decreases as $\theta$ decreases. For larger values of $p_H$ the reverse is true; $\xi_L$ increases as $\theta$ decreases.
4. Contrasts between cases: $\theta$ assumed known and assumed unknown

- For a given $p_D$, the ratio of var($Y_p$) with $\theta$ assumed known, relative to var($Y_p$) with $\theta$ assumed unknown, decreases as $p_H$ increases. Thus more can be gained from increasing acceleration when $\theta$ is assumed known than when $\theta$ is assumed unknown. As $P$ increases, however, this difference becomes smaller and the ratio of asymptotic variances becomes closer to 1.

- For the statistically optimum plan and $P < .01$, $\xi_L$ is much smaller for the test plans when $\theta$ is assumed unknown.

- For the optimized 4:2:1 allocation plan, $\xi_L$ is much smaller when $\theta$ is assumed unknown.

5. AN EXAMPLE

Meeker and Hahn (1985) illustrated their test plans with an example involving the design of an ALT for an adhesive-bonded power element. Here we present possible ALT plans for the same example but use the nonconstant $\sigma$ model. In this problem, the adhesive-bonded power element had been designed to have a lifetime of at least 10 years at a normal operating temperature of 50°C. The engineers who were going to run the experiment wanted to estimate the .10 quantile of the time-to-failure distribution at this design temperature, but the test had to be performed over a six-month period. The proportion failing at the design stress in six months was expected to be about .1%. Thus an ALT was suggested.

According to chemical reaction theory, the Arrhenius model is a physically appropriate model to relate testing at higher temperature levels to the design temperature for this product. The Arrhenius model is a linear relationship between the log time-to-failure and the reciprocal of absolute temperature. That is, $Y_i = \gamma_0 + \gamma_1 x_i + e_i$, where $Y_i = \log$ time-to-failure, $x_i = 1/(\text{°C} + 273)$, and $e_i$ is random error for the $i$th test unit. If time-to-failure follows a Weibull distribution, then $e_i$ follows a smallest extreme-value distribution with a location parameter 0 and a scale parameter $\sigma$. For $\sigma$ we assume the relationship $\log(\sigma_i) = \gamma_0^{[\sigma]} + \gamma_1^{[\sigma]} x_i$.

Three hundred units were available for testing. It was believed that approximately .1% of the test units would fail in six months at 50°C and that approximately 90% would fail at 120°C during that time period. Thus the “guess values” are $p_D = .001$ and $p_H = .90$. We present possible test plans for estimating the .10 quantile ($P = .1$) of the log time-to-failure distribution.

Table 5 presents test plans where $\theta$ is assumed to be known, either 1.0 or .8. For purposes of comparison, Table 6 shows test plans for the same parameter values, but assuming $\theta$ unknown. As expected, the statistically optimum plan in Table 5 ($\theta = 1.0$) is identical to the plan given by Meeker and Hahn (1985). Table 7 compares the asymptotic variances for each of the four cases. As expected the asymptotic variances are smaller for the optimum plans and when $\theta$ is assumed known. The compromise plans result in only a moderate loss of efficiency.

Figure 3 shows, for this particular example, the effect that different values of $\theta$ between .5 and 2.0 would have on the optimum values $\xi$ and $\tau$. (a) when $\theta$ is to be estimated and (b) when $\theta$ is assumed to be known. Over this range of $\theta$ (which is somewhat wider than .8 to 1.0 that we have seen in real data), changes in $\theta$ have little effect on the optimum $\tau$. For the case in which $\theta$ is to be estimated, smaller values of $\theta$ (implying that spread is relatively larger at lower levels of stress) call for the optimum plan to allocate more units to the lower level of stress, apparently to compensate for the larger spread. Particularly when $\theta$ is known, however, smaller values of $\theta$ call for the lower stress condition to move to

<table>
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<th>Standardized stress</th>
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<th>Numbers of test units allocated</th>
<th>Probability of failure</th>
<th>Expected numbers of test units failing</th>
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higher levels of stress where there will be less censoring and less spread in the failure-time distribution but less distance between the two levels of stress. Clearly, there are some complicated trade-offs involved in the optimization.

6. CONCLUDING REMARKS

This article has extended the theory for optimum accelerated test plans to the case in which the scale parameter $\theta$ is not constant. Optimum plans such as the ones presented here ignore some important practical considerations. The optimality depends on the assumed model. Meeker (1984) studied the effects of misspecifying the parameters ($p_D$ and $p_H$) for the constant $\sigma$ model and showed that the optimum plans are not robust to model inadequacies. Indeed, for tests that use only two levels of stress, there is no information to detect inadequacies in the assumed relationship between model parameters and stress. In spite of this, as suggested by Meeker and Hahn (1985), the optimum test plans do provide useful bounds on the best that one can do in a particular testing situation and important insights and a convenient starting point for designing more practical test plans, like the 4:2:1 plans.

Several areas remain for further research in this direction.

1. For the constant $\sigma$ model, Meeker and Hahn (1985) showed how to obtain compromise ALT plans that have reasonably good statistical precision but also meet important practical constraints. As shown by Meeker (1984), compromise plans also tend to be more robust to deviations from assumptions used in developing the test plans. We have shown that the 4:2:1 plans are reasonably efficient for the nonconstant $\sigma$ model. It might be possible, however, to find better compromise plans for this model.

2. Jensen and Meeker (1990) described a computer program that can be used to develop, evaluate, compare, and customize accelerated test plans. It would be useful to extend this software to allow evaluations for the nonconstant $\sigma$ model.

3. One difficulty with locally optimum designs is that they depend on the unknown model parameters. Meeker (1984) suggested a general method that can be used to assess the sensitivity of test plans to misspecification of the unknown inputs. The same ideas can be used for the current model. When actually planning an ALT, one should find plans over some plausible range of the inputs and use these to develop a compromise test plan. This process was systematized in a pseudo-Bayes framework by Chaloner and Larntz (1992). Their methods could also be applied to find test plans for the nonconstant $\sigma$ model.

4. Because they are based on large-sample asymptotic theory, the results presented in this article are approximate. In particular, the approximations are less accurate when the expected number of failures is small. Vander Wiel and Meeker (1990) compared coverage probabilities for confidence intervals for 4:2:1 ALT plans. These results give some insight into the adequacy of the asymptotic approximations for different values of $p_D$, $p_H$, $P$, and $n$. As shown in chapter 6 of Nelson (1990), Monte Carlo methods can be used to obtain better approximations of the properties of specified test plans.
5. In traditional experimental design with continuous factors, as well as in accelerated life testing, it has been traditional to assume that the experimental region is constrained. The constraints are typically chosen on the basis of subject-matter knowledge about the process under investigation. For example, in accelerated testing with temperature as the accelerating stress, it is common practice to limit temperature to a range over which the failure-causing process is accelerated according to relatively simple first-order kinetics so that the Arrhenius model will provide an adequate description of the temperature/life relationship. Meeker and Hahn (1985) emphasized that one should, in addition, limit the highest level of stress even more to minimize the inevitable extrapolation error. If one had a more general model for failure that would allow assessment of model bias, it would be possible to embed the optimization problem presented here into a larger optimization that also chooses the largest level of stress to minimize a more general criterion like mean squared error. Another alternative is to use Barton’s (1991) approach of choosing $x_H$ to be as small as possible, subject to meeting precision and sample-size constraints.

Along a similar line, we have assumed, as is often the case in practice, that the maximum length of the test is constrained. If this constraint is removed and if a cost-of-time function is specified, the length of the test could be optimized as well.

ACKNOWLEDGMENTS

We thank C. K. Chan for helping us understand the physical basis for some nonconstant $\sigma$ models and for providing references. Luis A. Escobar and Mike Hamada made helpful suggestions on a previous version of this manuscript. We would like to thank Vijay Nair, an associate editor, and two anonymous referees for making numerous helpful sug-
gestions that led to important improvements in the article. Much of Carol Meeter's work on this project was completed when she was a student at Iowa State University. Some of William Meeker's work was completed while he was a visitor with the Quality Process Center, AT&T Bell Laboratories, Holmdel, New Jersey.

APPENDIX: MAXIMUM LIKELIHOOD THEORY

This appendix presents the maximum likelihood theory for optimum ALT plans for nonconstant \( \sigma \). We present results for ALT models in which the log failure time \( Y \) follows a location-scale distribution with cdf

\[
Pr(Y \leq y) = \Phi\left(\frac{y - \mu}{\sigma}\right),
\]

where \( \mu \) and \( \sigma \) are location and scale parameters, respectively.

A.1 Log-Likelihood

Let the standardized log censoring time for the \( i \)th unit at the standardized stress level \( \xi_i \) be defined as

\[
\xi_i = \frac{\eta - \mu_i}{\sigma_i} = \frac{\eta - \beta_0 - \beta_1 \xi_i}{\sigma_i \theta_i}.
\]

Then \( p_i = \Phi(\xi_i) \) is the probability that the unit will fail before log time \( \eta \). Moreover, let

\[
Z_i = \frac{Y_i - \mu_i}{\sigma_i} = \frac{Y_i - \beta_0 - \beta_1 \xi_i}{\sigma_i \theta_i}
\]

be a standardized log failure time at stress level \( \xi_i \). Define the indicator function

\[
I_i = 1 \text{ if } Y_i \leq \eta, \text{ failure before log time } \eta
\]

\[
= 0 \text{ if } Y_i > \eta, \text{ censored at log time } \eta.
\]

Then the log-likelihood of a single observation at stress level \( \xi_i \) can be written as

\[
L_i = I_i \log \left[ \frac{\phi(Z_i)}{\sigma_i} \right] + (1 - I_i) \log[1 - \Phi(\xi_i)],
\]

where \( \phi = d\Phi/dz \).

The sample log-likelihood is the sum of the individual log-likelihoods over \( n \) independent observations. The log-likelihood is a function of \( Y_i, I_i, \xi_i, i = 1, n \) and the parameters \( \beta_0, \beta_1, \sigma_D, \) and \( \sigma_H \). The maximum likelihood estimates \( \hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}_D, \) and \( \hat{\sigma}_H \) are the parameter values that maximize the sample log-likelihood function.

A.2 Fisher Information and Asymptotic Covariance Matrices for the Constant \( \sigma \) Model

For the constant \( \sigma \) Weibull regression model (i.e., \( \theta = 1 \)), Nelson and Meeker (1978) showed that the Fisher information matrix for parameters \( \beta_0, \beta_1, \) and \( \sigma \), corresponding to the \( i \)th observation, can be expressed as

\[
F_i = \frac{1}{\sigma_i^2} \left[ \begin{array}{ccc} A(\xi_i) & \xi_i A(\xi_i) & B(\xi_i) \\ \xi_i A(\xi_i) & \xi_i^2 A(\xi_i) & \xi_i B(\xi_i) \\ B(\xi_i) & \xi_i B(\xi_i) & C(\xi_i) \end{array} \right].
\] (A.1)

The elements of this matrix are obtained from the negative expectations of the second-order partial derivatives of \( L_i \). That is,

\[
E\left( \frac{-\partial^2 L_i}{\partial \beta_j \partial \beta_k} \right) = \frac{\xi_j \xi_k}{\sigma_i^2} A(\xi_i), \quad j, k = 0, 1
\]

\[
E\left( \frac{-\partial^2 L_i}{\partial \beta_j \partial \sigma_D} \right) = \frac{\xi_j}{\sigma_i^2} B(\xi_i), \quad j = 0, 1
\]

\[
E\left( \frac{-\partial^2 L_i}{\partial \sigma_D^2} \right) = \frac{1}{\sigma_i^2} C(\xi_i),
\]

where \( \xi_{i0} = 1 \) and \( \xi_{i1} = \xi_i \).

The quantities \( A(\xi_i), B(\xi_i), \) and \( C(\xi_i) \) can be calculated for the normal (lognormal), smallest extreme-value (Weibull), and logistic (log-logistic) distributions using the computer algorithm of Escobar and Meeker (in press).

The Fisher information matrix for the entire experiment is calculated by summing the individual matrices, grouped by subexperiment:

\[
F_T = n \sum_{k=1}^{K} \pi_k F_k, \quad (A.2)
\]

where \( n \) is the sample size, \( K \) is the number of stress levels or subexperiments and \( \pi_k \) is the proportion of units allocated to the \( k \)th level. For a particular test plan, the asymptotic covariance matrix is the inverse of \( F_T \). We will show how to use these constant \( \sigma \) results to obtain the asymptotic covariance matrix for the nonconstant \( \sigma \) model in which \( \sigma = \sigma_D \theta_i \).

A.3 Fisher Information and Asymptotic Covariance Matrices for the Nonconstant \( \sigma \) Model

We can use standard derivative chain rules with the relationship \( \sigma = \sigma_D \theta_i^\gamma \) to express the second-order derivatives with respect to \( \sigma_D \) and \( \sigma_H \) for the nonconstant \( \sigma \) model as functions of the second-order derivatives for the constant \( \sigma \) model. In particular,

\[
\frac{\partial^2 L_i}{\partial \beta_j \partial \sigma_D} = \frac{(1 - \xi_i) \sigma_i}{\sigma_D} \frac{\partial^2 L_i}{\partial \beta_j \partial \sigma_D}, \quad j = 0, 1
\]

\[
\frac{\partial^2 L_i}{\partial \beta_j \partial \sigma_H} = \frac{-\xi_i \sigma_i}{\sigma_D \theta_i} \frac{\partial^2 L_i}{\partial \beta_j \partial \sigma_H}, \quad j = 0, 1
\]

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Noting that the expectation of the first derivatives is 0, that the chain rule coefficients of the second derivatives are constants, and that \( \sigma_i = \sigma_D \theta^{\xi_i} = \sigma_D \rho \) leads to

\[
F = \frac{1}{\sigma_D^2} \begin{bmatrix}
\frac{1}{\rho} A(\xi) & \frac{\beta_i}{\rho} A(\xi) & \frac{1}{\rho} B(\xi) & \frac{\beta_i}{\rho} B(\xi) \\
\frac{\beta_i}{\rho} A(\xi) & \frac{1}{\rho} A(\xi) & \frac{\beta_i}{\rho} B(\xi) & \frac{1}{\rho} B(\xi) \\
\frac{1}{\rho} B(\xi) & \frac{\beta_i}{\rho} B(\xi) & \frac{1}{\rho} C(\xi) & \frac{\beta_i}{\rho} C(\xi) \\
\frac{\beta_i}{\rho} C(\xi) & \frac{1}{\rho} C(\xi) & \frac{1}{\rho} C(\xi) & \frac{1}{\rho} C(\xi)
\end{bmatrix}
\]

The standardized failure time, \( \xi_i \), can be written as

\[
\xi_i = \frac{\eta - \mu_i}{\sigma_i} = \frac{\eta - \beta_0 - \beta_1 \xi_i}{\sigma_D \theta^{\xi_i}} = \frac{a - b \xi_i}{\theta^{\xi_i}},
\]

so the Fisher information matrix for the ith unit is a function of \( a, b, \theta, \) and \( \xi_i \).

The Fisher information matrix for the entire experiment is computed as in Equation (A.2), and the asymptotic covariance matrix \( \Sigma_\theta \) for \( (\beta_0, \beta_1, \sigma_D, \delta_H) \) is again the inverse of \( F_T \).

If \( \theta = \sigma_D \sigma_D \) is assumed known, there are three unknown parameters. The second-order partial derivatives with respect to \( \sigma_D \) for ith unit can again be expressed as functions of the second-order derivatives from the constant \( \sigma \) model by using derivative chain rules and the relationship \( \sigma_i = \sigma_D \theta^{\xi_i} \), with \( \theta \) constant. In particular,

\[
\frac{\partial^2 L_i}{\partial \beta_i \partial \sigma_D} = \frac{\sigma_i}{\sigma_D^2} \frac{\partial^2 L_i}{\partial \beta_i \partial \sigma_i}, \quad j = 0, 1
\]

\[
\frac{\partial^2 L_i}{\partial \sigma_D^2} = \frac{\sigma_i^2}{\sigma_D^2} \frac{\partial^2 L_i}{\partial \sigma_D^2}.
\]

Taking expectations, as in the model in which \( \theta \) is to be estimated, the Fisher information matrix is

\[
F_i = \frac{1}{\sigma_D^2} \begin{bmatrix}
\frac{1}{\rho_i} A(\xi_i) & \frac{\beta_i}{\rho_i} A(\xi_i) & \frac{1}{\rho_i} B(\xi_i) & \frac{\beta_i}{\rho_i} B(\xi_i) \\
\frac{\beta_i}{\rho_i} A(\xi_i) & \frac{1}{\rho_i} A(\xi_i) & \frac{\beta_i}{\rho_i} B(\xi_i) & \frac{1}{\rho_i} B(\xi_i) \\
\frac{1}{\rho_i} B(\xi_i) & \frac{\beta_i}{\rho_i} B(\xi_i) & \frac{1}{\rho_i} C(\xi_i) & \frac{\beta_i}{\rho_i} C(\xi_i) \\
\frac{\beta_i}{\rho_i} C(\xi_i) & \frac{1}{\rho_i} C(\xi_i) & \frac{1}{\rho_i} C(\xi_i) & \frac{1}{\rho_i} C(\xi_i)
\end{bmatrix}
\]

When \( \theta = 1 \), this reduces to the Fisher information matrix for the constant \( \sigma \) model in Equation (A.1), as expected. The asymptotic variance–covariance matrix \( \Sigma_\theta \) for \( (\beta_0, \beta_1, \sigma_D) \) is obtained by inverting \( F_T \), and \( F_T \) is obtained from the \( F_i \)'s, as in the \( \theta \) assumed unknown case.

A.4 Asymptotic Variances of Quantile Estimators

The maximum likelihood estimator of the \( P \) quantile of the log lifetime distribution at stress level \( \xi \) is \( \hat{Y}_p(\xi) = \hat{\beta}_0 + \hat{\beta}_1 \xi + u_p \sigma_D \theta^{\xi_i} \), where, for the smallest extreme-value distribution, \( u_p = \Phi^{-1}_\alpha(P) = \log[-\log(1 - P)] \). For the normal distribution, \( u_p = \Phi^{-1}_\alpha(P) \) is the inverse of the standard normal cdf. The asymptotic variance for this estimator is \( \text{var} (\hat{Y}_p(\xi)) = n \Sigma_\theta \nu' \), where \( \Sigma_\theta \) is the asymptotic variance–covariance matrix for \( (\beta_0, \beta_1, \sigma_D, \delta_H) \) and \( \nu = [1, \xi, u_p(1 - \xi)^{\theta^2}, u_p \theta^{\xi_i - 1}] \). At the design stress, when \( \xi = 0 \), this reduces to \( \text{var}(\hat{Y}_p(0)) = [1, 0, u_p, 0]' \Sigma_\theta [1, 0, u_p, 0]' \). When \( \theta \) is assumed known, the asymptotic variance has a simpler quadratic form—\( \text{var}(\hat{Y}_p(\xi)) = [1, \xi, u_p \theta^{\xi_i}] \Sigma_\theta [1, \xi, u_p \theta^{\xi_i}]' \), or when \( \xi = 0 \), \( \text{var}(\hat{Y}_p(0)) = [1, 0, u_p]' \Sigma_\theta [1, 0, u_p]' \) —at the design stress. In this case, \( \Sigma_\theta \) is the asymptotic variance–covariance matrix for \( (\beta_0, \beta_1, \delta_D) \). In either case, \( (n/\sigma_D^2) \text{var}(\hat{Y}_p(0)) \) is a function of \( P, \rho, \rho_H, \theta, \) and \( (\pi_b, \xi_i) \), \( k = 1, K \).

[Received July 1990. Revised April 1993.]

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