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This article provides a fast and accurate algorithm to compute the run-length probability distribution for cumulative sum charts to control process mean. This algorithm uses a fast and numerically stable recursive formula based on accurate Gaussian quadrature rules throughout the whole range of the computed run-length distribution and, therefore, improves the numerical efficiency and accuracy of existing methods. The algorithm may detect whether or not the geometric approximation is adequate and, when it is possible, it allows switching to the geometric recursion. The procedure may be applied not only to the normal distribution but also to nonsymmetric and long-tailed continuous distributions, some examples of which are provided. Methods to assess chart performance according to the run-length distribution, as well as some multivariate issues in statistical process control, are considered.

KEY WORDS: Cumulative sum charts; Decision limit; Gaussian quadrature; Integral equation; Multivariate statistics; Statistical process control.

1. INTRODUCTION

Cumulative sum (CUSUM) charts are used in statistical process control (SPC) to detect small shifts in a process mean. They have received considerable attention in the statistical literature; see, for example, Adams, Lowry, and Woodall (1992), Bissell (1969), Box and Luceño (1997, chap. 3), Brook and Evans (1972), Duncan (1986, chap. 22), Fellner (1990), Gan (1991, 1993), Goel and Wu (1971), Hawkins (1992), Hawkins and Olwell (1998), Kemp (1962). Lucas (1976), Lucas and Crosier (1982a, b), Montgomery (1996), Page (1954, 1961), Siegmund (1985), Vance (1986), Wadsworth, Stephens, and Godfrey (1986), Waldmann (1986), Woodall (1983, 1986), Woodall and Adams (1993), and Yashchin (1985). Suppose that a quality characteristic Y having mean \( \mu \) and standard deviation \( \sigma \) is being monitored so that small persistent deviations of \( \mu \) with respect to a target value \( T \) are detected as soon as possible.

Suppose also that samples of \( m \) independent observations of \( Y \) are taken and their averages \( \bar{y}_r \) are computed at equispaced times \( \tau D, \tau = 1, 2, \ldots \), where \( D \) is the time interval between successive samplings. We assume for simplicity, but without loss of generality, that \( T = 0 \) and \( \sigma m^{-1/2} = 1 \); otherwise, we simply standardize \( \bar{y}_r \). Then, starting with \( S_0 = 0 \) and using a reference value \( k \), one-sided CUSUM charts are obtained by plotting the cumulative sums

\[
S_\tau = \max\{S_{\tau-1} + (\bar{y}_\tau - k); 0\}, \quad \tau = 1, 2, \ldots, (1.1)
\]

versus \( \tau \). As soon as \( S_\tau \) reaches a decision interval \( h \), an alarm is triggered because the mean \( \mu \) might be larger than \( T \). Two-sided CUSUM charts to detect negative as well as positive deviations of \( \mu \) with respect to \( T \) are often implemented as a combination of two equivalent one-sided charts; namely, \( S^+_{\tau} = \max\{S^+_{\tau-1} + (\bar{y}_\tau - k^+); 0\}(\tau = 1, 2, \ldots) \) with decision interval \( h^+ > 0 \) and reference value \( k^+ \), which is intended to detect positive deviations and is equivalent to (1.1), and \( S^-_{\tau} = \max\{S^-_{\tau-1} + (\bar{y}_\tau - k^-); 0\}(\tau = 1, 2, \ldots) \) with decision interval \( h^- > 0 \) and reference value \( k^- \) designed to identify negative deviations of \( \mu \) with respect to \( T \).

The design of a one-sided CUSUM chart is usually based on an acceptable value of the mean when the process is in control, \( \mu_0 = T \), and a rejectable value of the mean, \( \mu_1 > T \), when the process is out of control. The parameters \( m, h, \) and \( k \) of the chart are selected to provide as few false alarms as possible when the process is in control and, simultaneously, swift detection when the process is out of control. Earlier attempts to assess the suitability of these parameters were based on error probabilities, as suggested by Johnson (1961); however, as shown by Adams et al. (1992), and using the words of Woodall and Adams (1993, p. 568), “the error probabilities required by Johnson’s method have no meaningful interpretation in terms of CUSUM chart performance” so that the adequacy of \( m, h, \) and \( k \) should nowadays be assessed in terms of average run lengths (ARL’s) or, preferably, in terms of the whole run-length probability distributions for acceptable and rejectable processes (Yashchin 1985, pp. 385–386). When the process is in control, the run-length distribution is often close to a geometric distribution, as shown, for example,
by Ewan and Kemp (1960) and Brook and Evans (1972), so that the standard deviation of the run length (SDRL) is almost as large as the ARL and the probability of early false alarms is relatively large. When the process is not in control, however, the run-length probability distribution may deviate considerably from a geometric distribution. The purpose of this article is to provide a fast and accurate algorithm to compute the run-length probability mass function (PMF) and cumulative distribution function (CDF) for one-sided CUSUM charts and to show new insights into the assessment of chart performance.

Section 2 provides the new algorithm. Section 3 gives computational details. Some multivariate issues are considered in Section 4.

2. EVALUATION OF THE RUN-LENGTH PROBABILITY DISTRIBUTION

Let $\Pr(n|z)$ be the probability that the decision interval $h$ is reached for the first time in exactly $n$ steps conditioned on the current value of the CUSUM statistic being $S_n = z$, where $0 \leq z < h$. Assume that the quality characteristic $Y$ is normally distributed, $T = 0$, $\sigma_m = 1$, and the averages in the sequence $\{\bar{y}_n\}$ follow independent and identically distributed (iid) normal distributions with mean $\mu$ and unit standard deviation. Let $\varphi(\cdot)$ and $\Phi(\cdot)$ be the probability density function (PDF) and CDF, respectively, for the standard normal distribution. Then

$$\Pr(1|z) = 1 - \Phi(h - z + k - \mu) \quad (2.1a)$$

and, for $n = 2, 3, \ldots$

$$\Pr(n|z) = \int_0^h \Pr(n - 1|x) \varphi(x - z + k - \mu) \, dx + \Pr(n - 1|0) \Phi(-z + k - \mu), \quad (2.1b)$$

which characterize the run-length probability distribution (Page 1954). The integral in (2.1b) may be computed very accurately and efficiently by using a Gauss-Legendre integration formula with $N$ points (e.g., see Press, Teukolsky, Vetterling, and Flannery 1992, sec. 4.5).

Let $\Omega$ be an $N$-dimensional square matrix whose element in the $i$th row and $j$th column is $\Omega_{ij} = \omega_j \varphi(x_j - x_i + k - \mu)$, and $\delta_1, \delta_2,$ and $p_n$ be $N$-dimensional column vectors whose elements in their $i$th rows are $\delta_{ij} = 1 - \Phi(h - x_i + k - \mu)$, $\delta_{2i} = \Phi(-x_i + k - \mu)$, and $p_{ni} = \Pr(n|x_i)$, respectively, where $(x_1, \ldots, x_N)$ and $(\omega_1, \ldots, \omega_N)$ are the Gauss-Legendre abscissas and weights for the range of integration $(0, h)$. Then, according to Gaussian quadrature rules, Equations (2.1) may be replaced by

$$p_1 = \delta_1 \quad (2.2a)$$

and

$$p_n = \Omega p_{n-1} + \delta_2 \Pr(n - 1|0), \quad (2.2b)$$

Similarly, particularization of Equations (2.1) for $z = 0$ gives

$$\Pr(1|0) = 1 - \Phi(h + k - \mu) \quad (2.3a)$$

and

$$\Pr(n|0) = \omega p_n + \Phi(k - \mu) \Pr(n - 1|0). \quad (2.3b)$$

where $\omega$ is an $N$-dimensional row vector whose element in its $i$th column is $\omega_j = \omega_j \varphi(x_j + k - \mu)$.

Equations (2.2) and (2.3) provide the basis for an efficient and accurate evaluation of the run-length probability distribution. Because $\Omega, \delta_1, \delta_2,$ and $\omega$ do not depend on $n$, they are evaluated only once. Then substitution of Equations (2.2) in Equation (2.3b) yields

$$\Pr(2|0) = \omega \delta_1 + \Phi(k - \mu) \Pr(1|0), \quad (2.4a)$$

and, for $n > 2$,

$$\Pr(n|0) = \omega \Omega^{n-2} \delta_1 + \omega \Omega^{n-3} \delta_2 \Pr(1|0) + \cdots + \omega \Omega^0 \delta_2 \Pr(n - 2|0) + \Phi(k - \mu) \Pr(n - 1|0). \quad (2.4b)$$

Equations (2.4) may be written as

$$\Pr(n|0) = \Phi(k - \mu) \Pr(n - 1|0) + \sum_{j=0}^{n-3} \omega_j \Pr(n - 2 - j|0) + k_1, n-2, \quad (2.5)$$

where the coefficients

$$k_{1j} = \omega \Omega^j \delta_1, \quad k_{2j} = \omega \Omega^j \delta_2, \quad j = 0, 1, \ldots, \quad (2.6)$$

do not depend on $n$ and hence are evaluated only once.

Alternative Formulation for the Coefficients $k_{1j}$ and $k_{2j}$.

Suppose that matrix $\Omega$ is diagonalized so that $\Omega = V \Lambda V^{-1}$, where the columns of matrix $V$ contain the eigenvectors of $\Omega$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$ contains the corresponding eigenvalues. Define the row vector $\bar{\omega} = \omega \cdot V$ and the column vectors $\bar{\delta}_1$, $\bar{\delta}_2$, and $\bar{\delta}_i$ be their respective $i$th elements. Because $\Omega^j = V \Lambda^j V^{-1}$, an alternative formulation for the coefficients in (2.6) is

$$k_{1j} = \sum_{i=1}^{N} \bar{\omega}_i \delta_{1i} \lambda_i^j, \quad k_{2j} = \sum_{i=1}^{N} \bar{\omega}_i \delta_{2i} \lambda_i^j, \quad j = 0, 1, \ldots \quad (2.7)$$

The eigenvalues of $\Omega$ must be smaller than 1 in absolute value for the sums in Equation (2.4b) or (2.5) to converge when $n$ increases so that their powers $\lambda_i^j$ and the coefficients $k_{1j}$ and $k_{2j}$ in (2.5) must tend to 0 geometrically as $j$ increases. (Extensive numerical evidence suggests that these eigenvalues are also real and nonnegative—at least under the normal assumption—and that the sequences $\{k_{1j}\}$ and $\{k_{2j}\}$ are monotonously decreasing with $j$.) Consequently, when $n - 3$ is larger than some $n^*$ such that $k_{2n^*}/k_{21}$ is negligible with respect to the numerical precision of the computer, Equation (2.5) may be replaced by

$$\Pr(n|0) = \Phi(k - \mu) \Pr(n - 1|0) + \sum_{j=0}^{n^*-1} k_{2j} \Pr(n - 2 - j|0) + k_{1, n-2}. \quad (2.8)$$
where the term $k_{1,n-2}$ is required only for small $n$'s. For large $M$, the number of operations required to evaluate the run-length distribution at $n = 1, \ldots, M$ using (2.7) is of order $Mn^*$, where $n^*$ is often about 30. The number of operations required to evaluate the coefficients $k_{1,n-2}$ and $k_{2,n}$ that are nonnegligible, which depends on the number of points $N$ used for the Gaussian quadrature, is always small and comparatively unimportant when $M$ is large.

**Geometric Approximation.** Equation (2.7) provides means to assess whether and when the geometric distribution is adequate as an approximation to the run-length distribution. Because the geometric distribution leads to the recursion

$$p_r(n|0) = \frac{1 - 1/\text{ARL}}{1} p_r(n-1|0),$$

we see that the drastic and general improvement in accuracy provided by Equation (2.7) over the geometric approximation is attained while keeping the computational effort at the same order of magnitude—that is, proportional to $M$. One way to decide whether the geometric approximation is adequate or not is to compute a lower quantile (e.g., the 1% quantile) of the true run-length distribution using (2.5), or (2.7), and to compare its value with the corresponding quantile for the geometric distribution having the same mean (ARL). This ARL may be easily evaluated with the method of Goel and Wu (1971)—that is, by solving with respect to $r_1$ and $r_2$ the linear systems $(I - \theta) r_1 = 1$ and $(I - \theta) r_2 = \delta_2$ so that, with $r_1(0) = \omega r_1 + 1$ and $r_2(0) = \omega r_2 + \Phi(k - \mu)$,

$$\text{ARL} = \frac{r_1(0)}{1 - r_2(0)}. \quad (2.9)$$

Clearly, when the ARL is large and the geometric approximation is appropriate, one may switch from (2.7) to (2.8) for $n$ larger than, say, the median run length to avoid some computations at the expense of introducing small errors in the computed upper quantiles of the run-length distribution (see Sec. 3). In the examples provided throughout the article, however, we never switch to the geometric recursion (2.8) so that the most accurate values provided by (2.5) and (2.7) are always shown.

**Some Related Results.** The ARL's corresponding to using the head start of Lucas and Crosier (1982a) may be obtained for the Gauss–Legendre abscissas $x_i$ as a by-product of the evaluation of the ARL $(0)$ in (2.9) considering that ARL $(x_i) = r_{1,i} + r_{2,i}$ ARL $(0)$, where $r_{1,i}$ and $r_{2,i}$ are the $i$th components of $r_1$ and $r_2$, respectively. The ARL$(z)$ can then be easily computed for any $z$ in the interval $(0, h)$. The methods in this section may also be applied (see Luceno 1999) to compute the ARL's and run-length probability distributions for some cumulative score (CUSCORE) charts considered by Box and Ramirez (1992) and Box and Luceno (1997, chap. 10).

**Example.** Table 1 provides the ARL and SDRL along with the median and some lower and upper quantiles of the run-length probability distribution for a few selected values of the standardized parameters $\mu - k$ and $h$. The numbers given in the table should be interpreted as the factors of the sampling interval $D$ that provide the corresponding distribution characteristics; thus, if $D$ were equal to one hour, then the numbers in the table would give the ARL, SDRL, median, and quantiles in hours. We note that, even when the ARL is large, the lower quantiles of the distribution may be remarkably low. This means that, even when the rate of false alarms is small, a relatively large percent of false alarms occur very early. For example, suppose that a CUSUM chart with $D = 10$ miles, $k = 1.5$, and $h = 2.5$ were used to diagnose malfunctioning of the exhaust system of a kind of cars (note that we use mileage rather than time to measure the sampling interval); this scheme would produce an ARL = 108,613 miles for $\mu = 0$, giving the impression that most cars would not have any false alarms in their useful lives; nonetheless, according to Table 1, 1% (or 5%) of the false alarms would occur not later than 1,110 (or 5,580) miles so that many new cars would suffer false alarms.

### Table 2. ARL, SDRL, Median, and 1%, 5%, and 25% Lower and Upper Quantiles of the Run-Length Distribution for a CUSUM Chart With Some Selected Values of the Standardized Parameters $\mu - k$ and $h$

<table>
<thead>
<tr>
<th>$\mu - k$</th>
<th>$h$</th>
<th>ARL</th>
<th>SDRL</th>
<th>1%</th>
<th>5%</th>
<th>25%</th>
<th>Median</th>
<th>75%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.5</td>
<td>2.5</td>
<td>10,051.3</td>
<td>10,050.9</td>
<td>111</td>
<td>569</td>
<td>3,126</td>
<td>7,520</td>
<td>16,066</td>
<td>32,536</td>
<td>60,013</td>
</tr>
<tr>
<td>.5</td>
<td>2.5</td>
<td>5,4228</td>
<td>3,3786</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>-1.5</td>
<td>1.5</td>
<td>549.69</td>
<td>548.90</td>
<td>6</td>
<td>29</td>
<td>159</td>
<td>381</td>
<td>762</td>
<td>1,645</td>
<td>2,529</td>
</tr>
<tr>
<td>.5</td>
<td>1.5</td>
<td>3,5009</td>
<td>2,3694</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

Note: The numbers given in the table are to be interpreted as factors of the sampling interval $D$—that is, 10 miles in the example of Section 2.
Table 3. ARL, SDRL, Median, and 1%, 5%, and 25% Lower and Upper Quantiles of the Run-Length Distribution for a CUSUM Chart
With Some Selected Values of the Standardized Parameters \( \mu - k \) and \( h \), With Corresponding Quantiles for the
Geometric Distribution Having the Same Mean (in parentheses)

<table>
<thead>
<tr>
<th>( \mu - k )</th>
<th>( h )</th>
<th>ARL</th>
<th>SDRL</th>
<th>1%</th>
<th>5%</th>
<th>25%</th>
<th>Median</th>
<th>75%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.25</td>
<td>11.8</td>
<td>5,164.94</td>
<td>5,136.28</td>
<td>80</td>
<td>292</td>
<td>1,506</td>
<td>3,589</td>
<td>7,149</td>
<td>15,416</td>
<td>23,682</td>
</tr>
<tr>
<td>0</td>
<td>60.0</td>
<td>3,741.18</td>
<td>3,064.66</td>
<td>475</td>
<td>746</td>
<td>1,500</td>
<td>2,634</td>
<td>4,036</td>
<td>6,017</td>
<td>14,698</td>
</tr>
</tbody>
</table>

Legendre abscessas and weights, and solutions to linear systems. The execution time on a portable computer for evaluating the run-length PMF and CDF in as many as one million points is shorter than half a minute. As recommended by Goel and Wu (1971), \( N = 15 \) Gaussian points may often suffice, but we prefer a moderately larger \( N \) because this may improve overall accuracy and affects the total execution time only marginally (we have used up to 140 points). The algorithm is particularly convenient for evaluating lower quantiles of the run-length distribution because the computation proceeds from \( n = 1 \) onward and hence may be stopped as soon as the larger quantile to be computed is reached and also because one may switch from the recursive formula (2.7) to the geometric approximation (2.8) when the latter is adequate (which may be assessed as indicated in Sec. 2).

Examples. To assess the accuracy of the computations, we use first a very extreme example of a CUSUM chart—namely, the second example of Woodall (1983, p. 298). The parameter values are \( \mu - k = -1.4 \) and \( h = 4 \). The corresponding ARL obtained using double-precision arithmetic and the method of Goel and Wu (1971) is 421,705.34. We have run our computer program to evaluate the run-length probability distribution from \( n = 1 \) to \( n = 2^{23} \) (more than eight million points), and we have reevaluated the ARL using this PMF, the value resulting with the algorithm in Section 2 is 421,705.40, which shows a relative error of \( 1.4 \times 10^{-7} \).

Table 2 provides the ARL, SDRL, median, and lower and upper quantiles of the run-length distribution along with the corresponding values for the geometric distribution with mean 421,705.40. One can see that both distributions are very close but not identical, the run-length distribution appears to be slightly less spread than the geometric distribution, as suggested by both the SDRL and the lower and upper quantiles. We note that the values in Table 2 are more accurate than those provided by Woodall (1983) because Equation (2.7) allows us to use accurate Gaussian quadrature rules throughout the whole range of values of \( n \), whereas Woodall approximated the ARL and quantiles by applying Simpson's rule only for \( n = 1, \ldots, 25 \), and also because our methods are very stable numerically.

Table 3 shows two additional examples using very large values of \( h \). We can see that the ARL's are relatively large but, nevertheless, the geometric approximation (in parentheses) is far from adequate. Even though these large values of \( h \) are hardly ever used in practice, our numerical methods continue to be very accurate and stable provided the number \( N \) of Gauss-Legendre points is taken relatively large. Figure 1 shows the ARL \( (z) \), \( 0 < z < h \), corresponding to using the head start of Lucas and Crosier (1982a) for one of these examples.

**Figure 1.** ARL \( (z) \), \( 0 < z < h \), Corresponding to Using the Head Start of Lucas and Crosier (1982a) for \( \mu - k = \pm 0.25 \) and \( h = 11.8 \).
distribution, the smaller the ARL appears to be for the same $\mu - k$ and $h$. (A normalizing transformation might be useful to alleviate this problem; this transformation might be obtained by equating the standard normal CDF to the true underlying CDF if this is known at least approximately and should be applied to each observation before updating the CUSUM statistic.)

As an additional example, suppose that $h$ is unknown and we replace the definition of the CUSUM statistic in (1.1) by $S_T = \max\{S_{T-1} + (y_{T} - k)/\sqrt{\text{df}}, 0\}$, where, for $\tau = 1, 2, \ldots, \{y_{T}, s_{T}\}$ are iid sample averages and standard deviations. Then the standard normal distribution should be replaced by the noncentral Student-t distribution with noncentrality parameter $(\mu - k)/\sqrt{\text{df}}$ and $v$ (possibly $m - 1$) df. Computer routines for the noncentral Student-t PDF and CDF can be found, for example, in Matlab (function “nctpdf”) and Lenth (1989), respectively (see also Wilks 1962, p. 247; Owen 1968). Table 5 gives the ARL, SDRL, and quantiles of the run-length distribution for $(\mu - k)/\sqrt{\text{df}} = \pm 1.4, h/\sqrt{\text{df}} = 3$, and several values of $v$. One can see that the effect of the underlying distribution on the run-length distribution is much greater when the process is in control than when the process is out of control.

### 4. SOME MULTIVARIATE ISSUES

We have assumed so far that the values in the sequence $\{y_{T}\}$, which are used to build the CUSUM chart, are calculated as averages of $m$ simultaneous observations taken on a quality characteristic $Y$. The only hypotheses underlying the methods in the article, however, are that $y_{1}, y_{2}, \ldots$ are iid random variables. Therefore, nothing can prevent us from applying the same charts and methods to other sequences of relevant statistics calculated from simultaneous observations of available process variables and/or quality characteristics, provided the sampling interval is large enough so that the basic Shewhart–Deming assumption of independence is satisfied (or, otherwise, adequate prewhitening is applied; see Johnson and Bagshaw 1974; Bagshaw and Johnson 1975; Luceno and Box 2000), and the normal distribution is replaced by the appropriate distribution corresponding to the statistic being used; care should be taken, however, in ascertaining this distribution because, as we showed in Section 3 and Tables 4 and 5, the ARL may change substantially depending on the shape of the underlying distribution. In particular, CUSUM charts may be run in which the values $y_{T}$ in the sequence represent a principal component resulting from a principal-component analysis (PCA) or a latent variable resulting from a projection to latent structure (PLS) analysis, or the values of a Hotelling’s $T^2$ statistic, or a robust estimate of location, to mention just a few possibilities (see Wold, Esbensen, and Geladi 1987; MacGregor 1997; Hawkins and Olwell 1998, chap. 8). Proceeding in this form, many standard SPC methods may easily be adapted to multivariate situations. The main challenge in a practical situation seems, therefore, to rely on the search for relevant statistics to be considered, which might not be of the linear forms usually considered in PCA or PLS, and their distributions.

### Table 4. ARL, SDRL, Median, and 1%, 5%, and 25% Lower and Upper Quantiles of the Run-Length Distribution for a CUSUM Chart With Standardized Parameters $\mu - k = -1.4$ and $h = 2(5)$, Assuming the Underlying Distribution to Be Either Normal or Gumbel

<table>
<thead>
<tr>
<th>Underlying distribution</th>
<th>$h$</th>
<th>ARL</th>
<th>SDRL</th>
<th>1%</th>
<th>5%</th>
<th>25%</th>
<th>Median</th>
<th>75%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>4.0</td>
<td>421.705.4</td>
<td>421.705.4</td>
<td>4.241</td>
<td>21.633</td>
<td>121.319</td>
<td>292.305</td>
<td>584.607</td>
<td>1,263.311</td>
<td>1,942.016</td>
</tr>
<tr>
<td>Gumbel</td>
<td>4.0</td>
<td>1,181.7</td>
<td>1,180.7</td>
<td>13</td>
<td>62</td>
<td>341</td>
<td>819</td>
<td>1,638</td>
<td>3,538</td>
<td>5,438</td>
</tr>
<tr>
<td>Normal</td>
<td>3.5</td>
<td>104,379.4</td>
<td>104,377.1</td>
<td>1.051</td>
<td>5.256</td>
<td>30.030</td>
<td>72.351</td>
<td>144.700</td>
<td>312.688</td>
<td>480.677</td>
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<tr>
<td>Gumbel</td>
<td>3.5</td>
<td>668.5</td>
<td>667.9</td>
<td>8</td>
<td>35</td>
<td>193</td>
<td>404</td>
<td>927</td>
<td>2,002</td>
<td>3,077</td>
</tr>
<tr>
<td>Normal</td>
<td>3.0</td>
<td>25,668.8</td>
<td>25,666.9</td>
<td>260</td>
<td>1,318</td>
<td>7,386</td>
<td>17,793</td>
<td>35,584</td>
<td>76,693</td>
<td>118,203</td>
</tr>
<tr>
<td>Gumbel</td>
<td>3.0</td>
<td>378.0</td>
<td>377.1</td>
<td>5</td>
<td>20</td>
<td>109</td>
<td>262</td>
<td>524</td>
<td>1,131</td>
<td>1,738</td>
</tr>
<tr>
<td>Normal</td>
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<td>6,220.4</td>
<td>6,219.0</td>
<td>64</td>
<td>320</td>
<td>1,791</td>
<td>4,312</td>
<td>8,623</td>
<td>16,932</td>
<td>28,611</td>
</tr>
<tr>
<td>Gumbel</td>
<td>2.5</td>
<td>213.1</td>
<td>212.4</td>
<td>3</td>
<td>12</td>
<td>62</td>
<td>148</td>
<td>358</td>
<td>549</td>
<td>979</td>
</tr>
<tr>
<td>Normal</td>
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<td>1,605.2</td>
<td>1,504.1</td>
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<td>1,043</td>
<td>2,086</td>
<td>4,508</td>
<td>6,930</td>
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<td>119.1</td>
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<td>35</td>
<td>83</td>
<td>166</td>
<td>358</td>
<td>549</td>
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</tbody>
</table>

### Table 5. ARL, SDRL, Median, and 1%, 5%, and 25% Lower and Upper Quantiles of the Run-Length Distribution for a CUSUM Chart With Standardized Parameters $(\mu - k)/\sqrt{\text{df}} = \pm 1.4$ and Several Values of the Degrees of Freedom

<table>
<thead>
<tr>
<th>Noncentrality parameter</th>
<th>Degrees of freedom</th>
<th>ARL</th>
<th>SDRL</th>
<th>1%</th>
<th>5%</th>
<th>25%</th>
<th>Median</th>
<th>75%</th>
<th>95%</th>
<th>99%</th>
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<td>633.2</td>
<td>632.5</td>
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<td>439</td>
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<td>2,914</td>
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<td>477</td>
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<td>7,620</td>
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<td>266</td>
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<td>3,583</td>
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<td>23,799</td>
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<td>14,664.2</td>
<td>146</td>
<td>754</td>
<td>4,220</td>
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<td>20,330</td>
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REFERENCES


