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TWO MEASURES OF INCOHERENCE: HOW NOT TO GAMBLE IF YOU MUST

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The degree of incoherence, when previsions are not made in accordance with a probability measure, is measured by either of two rates at which an incoherent bookie can be made a sure loser. Each bet is considered as an investment from the points of view of both the bookie and a gambler who takes the bet. From each viewpoint, we define an amount invested (or escrowed) for each bet, and the sure loss for incoherent previsions is divided by the escrow to determine the rate of incoherence. Potential applications include the treatment of arbitrage opportunities in financial markets and the degree of incoherence of classical statistical procedures. We illustrate the latter with the example of hypothesis testing at a fixed size.

1. Introduction. One of the most compelling arguments for modeling uncertainty with probability is the "Dutch Book" argument, which says that if you are willing to accept either side of each bet implied by your statements, either

(a) those statements are "coherent," that is they comport with the axioms of probability, or

(b) a gambler betting against you can choose bets so that you are a sure loser.

Excellent introductions to the concepts of coherence and Dutch Book can be found in Shimony (1955), Freedman and Purves (1969) and de Finetti (1974, Section 3.3).

Similarly, financial analysts search the markets for arbitrage opportunities, playing the role of the gambler. As a practical matter, it is very difficult to structure one's statements of probabilities (i.e. previsions) in such a way that they both reflect one's beliefs and are coherent (see Kadane and Wolfson, 1998). Yet the dichotomy above does not allow for discussion of what sets of previsions may be "very" incoherent or just "slightly" incoherent. This paper explores a remedy for this by studying how quickly an incoherent bookie can be forced to lose money. A faster rate of sure financial decline is associated with a greater degree of incoherence.

The problem as stated so far requires some normalization. Suppose that a particular combination of gambles yields a sure loss y for the bookie. Then multiplying each gamble by the same constant k will create a combination of gambles that yields sure loss ky. In this paper we explore two normalizations, one from the gambler's perspective and one from the bookie's perspective. The bookie obviously cannot be assumed to have infinite resources. A wise gambler would want to be sure that the bookie could cover all the bets.

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One way to do this would be to require the bookie to escrow the maximum amount that the bookie could lose. Thus we can ask how much the bookie can be forced to lose for sure, given a specified level of escrow that the bookie can offer. This is the subject of Sections 2 and 5.1.

Alternatively, thinking of the financial market as the bookie and an individual investor as the gambler, it is the gambler's resources that are limited. Then one can ask how much the gambler can be guaranteed to win, for a given level of gambler escrow. This, in turn, is studied in Sections 4 and 5.2. This sense of escrow corresponds to the purchase of securities in which an investor might lose an amount as large as the entire purchase price, but no more than that. In Section 7, we show how our results apply equally to the case in which coherence is defined in terms of quadratic scoring rules as in de Finetti (1974, Section 3.3.6).

2. The bookie's perspective. Think of a random variable X as a function from some space T of possibilities to the real numbers. We assume that, for a bounded random variable X, a bookie might announce some values of x such that he/she finds acceptable all gambles whose net payoff to the bookie is $\alpha(X-y)$ for $\alpha > 0$ and $y < x$. Each such x will be called a *lower prevision for* X. In addition, or alternatively, the bookie may announce some values of x such that the gamble $\alpha(X-y)$ is acceptable when $\alpha < 0$ and $y > x$. These x will be called *upper previsions for* X. We allow that the bookie might announce only upper previsions or only lower previsions or both. For example, if X is the indicator I_A of an event A the bookie might announce that he/she finds acceptable all gambles of the form $\alpha(I_A - x)$ for $x < p$ if $\alpha > 0$ but no other gambles, in particular, not for $x = p$. It will turn out not to matter for any of our results whether or not the bookie finds the gamble $\alpha(I_A - p)$ acceptable. In the special case in which x is both an upper prevision and a lower prevision, we call x a *prevision* of X and denote it $P(X)$.

It will be convenient to assume that, whenever both an upper prevision x^+ and a lower prevision x^- have been assessed for the same random variable X, $x^- \le x^+$, otherwise the bookie is willing to sell X for a certain price and then buy it right back for a higher price. Although such incoherence could be measured, it requires cumbersome bookkeeping that makes general results difficult to understand. See Example 4 for an example of what can happen without this assumption. In particular, this assumption implies that there can be at most one prevision of X.

A gambler who considers whether or not to place a bet like $\alpha(X-x)$ with a bookie would like some sort of guarantee that the bookie would be able to pay in the event that $\alpha(X-x) < 0$. If the bookie takes several bets $\{\alpha_i(X_i - x_i)\}_{i=1}^n$, each possibly from a different gambler, then each gambler wants a guarantee that the bookie can pay up. For this reason, we introduce an *escrow account* into which the bookie must pay the maximum amount that he/she might lose for every gamble that he/she is making. The maximum amount that a bookie can lose for the gamble $\alpha(X-x)$ is $-\min\{0, \inf_{t\in T} \alpha(X(t)-x)\}$ since the bookie cannot lose if $\inf_{t\in T} \alpha(X(t)-x) > 0$. The escrow account is returned to the bookie at the time he/she must settle all the bets.

Example 1. Let A be an arbitrary event which is neither certain to occur nor certain to fail. Suppose that a lower prevision p is given, and consider the gamble $\alpha(I_A - p)$ with $\alpha > 0$. Here, $X(t) = I_A(t)$, the indicator of whether or not $t \in A$. Then $\inf_t \alpha(I_A(t) - p) = -\alpha p$, and the necessary escrow is αp. If an upper prevision q is given and $\alpha < 0$, then $\inf_t \alpha(I_A(t) - q) = \alpha(1-q)$, and the necessary escrow is $-\alpha(1-q)$.

A collection $x_1, \ldots, x_n$ of upper and/or lower previsions for $X_1, \ldots, X_n$ respectively is *incoherent* if there exists $\epsilon > 0$ and a collection of acceptable gambles $\{\alpha_i(X_i - y_i)\}_{i=1}^n$ such that

$$\sup_{t \in T} \sum_{i=1}^n \alpha_i(X_i(t) - y_i) < -\epsilon, \tag{2.1}$$

in which case we say that a *Dutch Book* has been made against the bookie. Of course, $\alpha_i > 0$ and $y_i < x_i$ if x_i is a lower prevision for X_i and $\alpha_i < 0$ and $y_i > x_i$ if x_i is an upper prevision for X_i. For a combination of gambles $Y = \sum_{i=1}^n \alpha_i(X_i - y_i)$, define the *guaranteed loss* to be $G(Y) = -\min\{0, \sup_{t \in T} Y(t)\}$. So, Dutch Book can be made if there exists a combination of acceptable gambles whose guaranteed loss is positive. When a Dutch Book is made, the larger the absolute values of the α_i are, the larger the guaranteed loss will be and the more the incoherent bookie can be forced to lose. We want to find out how much an incoherent bookie can be forced to lose per unit paid into the escrow account. Define ρ, the *extent of incoherence* or *maximum rate of guaranteed loss*, to be the supremum, over all combinations Y of acceptable gambles, of $G(Y)$ subject to the escrow for Y being at most 1. If the previsions are incoherent, then $\rho > 0$. If the previsions are coherent, then $\rho = 0$. There is a slightly simpler way to compute ρ than directly from the definition.

THEOREM 1. *Let $x_1, \ldots, x_n$ be a collection of incoherent upper and/or lower previsions for $X_1, \ldots, X_n$. Define*

$$\begin{aligned} g(\alpha_1, \ldots, \alpha_n) &= \sup_{t \in T} \sum_{i=1}^n \alpha_i(X_i(t) - x_i), \\ h(\alpha_1, \ldots, \alpha_n) &= -\sum_{i=1}^n \min\{0, \inf_{t \in T} \alpha_i(X_i(t) - x_i)\}. \end{aligned}$$

Then

$$\rho = \sup_{\alpha_1, \ldots, \alpha_n} \frac{-g(\alpha_1, \ldots, \alpha_n)}{h(\alpha_1, \ldots, \alpha_n)}. \tag{2.2}$$

As with all of the more lengthy proofs in this paper, the proof of Theorem 1 is in an appendix. Theorem 1 allows us to ignore the fact that the gamble $\alpha(X - x)$ might not be acceptable when x is a lower or upper prevision for X if we are proving results concerning the maximum rate of guaranteed loss. This consideration leads to the following equivalent form of (2.2):

$$\rho = -\inf_{\alpha_1, \ldots, \alpha_n} g(\alpha_1, \ldots, \alpha_n), \text{subject to } h(\alpha_1, \ldots, \alpha_n) \leq 1. \tag{2.3}$$

The infimum is taken over those α_i that have the appropriate signs.

Note that if a collection of gambles satisfies the escrow condition $h(\alpha_1, \ldots, \alpha_n) \leq 1$, then every subcollection also satisfies the escrow condition because the escrow amount is nonnegative for each gamble and the escrow amounts for the different gambles are computed independently of each other and add up to the value of h. Also, since the escrow is the most the bookie could lose, we have $\rho \leq 1$ in all cases.

EXAMPLE 2. Consider the case of $X(t) = c$, a constant, for all t. For a gamble $\alpha(X - p)$, $\inf_t \alpha(X(t) - p) = (c - p)\alpha$, since this is the only possible value. The necessary escrow is 0 if $(c - p)\alpha \geq 0$ and $(p - c)\alpha$ if $(c - p)\alpha < 0$. In this example, $g(\alpha) = (c - p)\alpha$ for all α. If either a lower prevision $p > c$ or an upper prevision $p < c$, then $-g(\alpha) = h(\alpha)$ for acceptable gambles, which makes $\rho = 1$.

As Example 2 illustrates, constant random variables are very special. If $X = c$ is constant, and the bookie assigns an upper and/or lower prevision p on the correct side(s) of c, then the gamble $\alpha(X-p)$ is a constant nonnegative number whenever it is acceptable. On the other hand, if the bookie assigns an upper or lower prevision on the wrong side of c, no matter how close to c, he/she will lose the entire escrow for sure. In fact, if a collection of previsions includes even one incorrect upper or lower prevision for a constant, then $\rho = 1$ is obtained by ignoring all the other previsions and placing the appropriate bet on the constant. This all-or-nothing behavior occurs only with constant random variables, as we illustrate next.

Example 3. Let X take at least two different values, and let $x^m = \inf_t X(t)$ and $x^M = \sup_t X(t)$. Suppose that $p^- > x^M$ is an incoherent lower prevision for X. If $\alpha > 0$, the escrow for gamble $\alpha(X-p^-)$ is $\alpha(p^- - x^m)$. This implies that $\rho = (p^- - x^M)/(p^- - x^m) < 1$. Similarly, if $p^+ < x^m$ is an incoherent upper prevision and $\alpha < 0$, the escrow is $\alpha(x^M - p^+)$ and $\rho = (x^m - p^+)/(x^M - p^+) < 1$. Note also that $\rho < 1$ in these cases no matter how far p^+ and/or p^- are away from the interval $[x^m, x^M]$. Contrast this with the situation in Example 2 in which $\rho = 1$ no matter how close the incoherent previsions are to c.

Because the case of constant random variables is so special and is handled already, we deal only with nonconstant random variables for the remainder of this paper. In Section 4, we introduce a different type of escrow which also leads to the conclusion that all out-of-range previsions, i.e., upper prevision less than $\inf_t X(t)$ or lower prevision greater than $\sup_t X(t)$, are equally incoherent.

The following example illustrates why it is convenient to assume that upper previsions are always at least as large as lower previsions.

Example 4. Let A_1, A_2, A_3 be a partition of T and let $p_1 = 0.5$, $p_2 = 0.4$, and $p_3 = 0.4$ be lower previsions for the three events. Let $q_1 < 0.5$ be an upper prevision for A_1. The most general combination of gambles to consider is

$$\alpha_1(I_{A_1} - 0.5) + \alpha_2(I_{A_2} - 0.5) + \alpha_3(I_{A_3} - 0.4) - \alpha_4(I_{A_1} - q_1),$$

with all $\alpha_i \geq 0$. The escrow for this combination is

$$0.5\alpha_1 + 0.4\alpha_2 + 0.4\alpha_3 + (1 - q_1)\alpha_4. \tag{2.4}$$

Subject to (2.4) equaling 1, the negative of (2.2) becomes

$$\max\{\alpha_1, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4\} - 1, \tag{2.5}$$

which we can now try to minimize. It is not difficult to see that we need $\alpha_1 = \alpha_2 + \alpha_4 = \alpha_3 + \alpha_4$ in order to minimize (2.5). Combining these equations with (2.4) gives

$$\alpha_2 = \alpha_3 = \frac{1 - (1.5 - q_1)\alpha_1}{q_1 - 0.2}, \quad \alpha_4 = \frac{1.3\alpha_1 - 1}{q_1 - 0.2},$$

and we are trying to minimize α_1 subject to all four $\alpha_i \geq 0$. If $q_1 < 0.2$, we need $\alpha_1 = \alpha_4 = 1/(1.5 - q_1)$ and $\alpha_2 = \alpha_3 = 0$. This corresponds to gambling only using the upper and lower previsions on A_1. If $q_2 > 0.2$, we need $\alpha_1 = \alpha_2 = \alpha_3 = 1/1.3$ and $\alpha_4 = 0$. this corresponds to ignoring the upper prevision on A_1 altogether. For cases with more than three events and more than one upper prevision less than the corresponding lower prevision, we must solve a general linear programming problem.

For proving theoretical results, it is convenient to be able to assume that upper and lower previsions have been announced for all random quantities under consideration. Fortunately, it is possible to do this without requiring the bookie to accept gambles that might be considered unfavorable. Suppose that x_+ is an upper prevision for X. Let $x_- = \min\{x_+, \inf_t X(t)\}$. Suppose that we pretend as if x_- is a lower prevision for X. The gambles that this commits the bookie to accept are $\alpha(X - y)$ for $\alpha > 0$ and $y < x_-$. For all t, $\alpha(X(t) - y) > 0$, so the bookie will not have to escrow for such a gamble and he/she will never lose this gamble. Hence, it cannot contribute to the degree of incoherence if we assume that x_- is a lower prevision for X. Similarly, if x_- is an announced lower prevision, then $x_+ = \max\{x_-, \sup_t X(t)\}$ can be taken to be an upper prevision for X without loss of generality.

There are a few other upper or lower previsions that can be inferred from those announced. For example, if x_- is an announced lower prevision for X, let $Y = c - X$ for some constant c and let $y_+ = c - x_-$. Since $\alpha(X(t) - [x_- - \epsilon]) = -\alpha(Y(t) - [y_+ + \epsilon])$ for all t, we have that y_+ is an upper prevision for Y even if the bookie did not realize it. Also, the escrow is the same for both sides of this equation because the values are equal for all t. Similarly, if x_+ is an announced upper prevision for X then $c - x_+$ is a lower prevision for $c - X$. For the case in which $X = I_A$, the indicator of an event A, we can let $c = 1$ and we have that p_- is a lower prevision for A if and only if $1 - p_-$ is an upper prevision for A^C. There is an interesting extension of this to disjoint events.

RESULT 1. *Let $A_1, \ldots, A_m$ be disjoint events with lower previsions $p_1, \ldots, p_m \geq 0$. Let $B = (\cup_{i=1}^m A_i)^C$. Then, the value of ρ is unchanged if we act as if if $q = 1 - \sum_{i=1}^m p_i$ were an upper prevision for B.*

PROOF. For $\alpha < 0$, write

$$\begin{aligned} \alpha(I_B - [q + \epsilon]) &= \alpha\left(1 - \sum_{i=1}^m I_{A_i} - 1 + \sum_{i=1}^m \left[p_i - \frac{\epsilon}{m}\right]\right) \\ &= \sum_{i=1}^m -\alpha\left(I_{A_i} - \left[p_i - \frac{\epsilon}{m}\right]\right), \end{aligned} \tag{2.6}$$

which is acceptable since $-\alpha > 0$ and the p_i are lower previsions. The escrow for the far left-hand side of (2.6) is $-\alpha(1 - q - \epsilon) = -\sum_{i=1}^m \alpha[p_i - \epsilon/m]$, which is the same as the escrow for the far right-hand side. Hence, even if the bookie does not realize it, he/she is willing to accept all gambles of the form $\alpha(I_B - y)$ for $\alpha > 0$ and $y < p$, hence p is an lower prevision for B. □

The situation described in this section is similar to the one discussed by Nau (1989, 1992). The author of those papers allows, in addition, that a *confidence weight* accompanies each upper or lower prevision, to specify the amount of money that a bookie is willing to risk at specific odds. Our concept of maximum rate of guaranteed loss is a special case of *relative incoherence* from Nau (1989, p. 389), when the confidence weights are all equal. Instances of our Lemma 1 and (6.4), for example, can be found by applying Nau's algorithms. Likewise, the topic of Section 6, conditional previsions, is raised by Nau (1989). By contrast, we offer analytic forms for optimal strategies, including applications to hypothesis testing. We also define and explore the gambler's perspective (*maximum rate of guaranteed profit*), and we relate our theory to proper scoring rules as well as to gambling.

For the case in which the collection of gambles concerns a partition of the space of possibilities, we have the following result.

THEOREM 2. *Let $\{A_i\}_{i=1}^n$ be a finite partition of the set T of possibilities, with $n > 1$ and all A_i nonempty. Let $q_1, \ldots, q_n$ be upper previsions and let $p_1, \ldots, p_n \geq 0$ be lower previsions for the partition elements. Let $s^+ = \sum_{i=1}^n q_i$ and let $s^- = \sum_{i=1}^n p_i$.*

1. *If $s^- > 1$, then $\rho = (s^- - 1)/s^-$ and $\alpha_j = 1/s^-$ for all j achieves the* inf *in (2.3). If $0 < p_i < 1$ for all i, then the $\alpha_1, \ldots, \alpha_n$ that achieve the* inf *in (2.3) are unique.*

2. *If $s^+ < 1$, then $\rho = (1 - s^+)/(n - s^+)$ and $\alpha_j = -1/(n - s^+)$ for all j achieves the* inf *in (2.3). If $0 < q_i < 1$ for all i, then the $\alpha_1, \ldots, \alpha_n$ that achieve the* inf *in (2.3) are unique.*

Note that ρ is strictly increasing in s^- for $s^- > 1$ and is strictly decreasing in s^+ for $s^+ < 1$. Since lower previsions are never larger than upper previsions, it is easy to see that it is impossible for both $s^- > 1$ and $s^+ < 1$.

For completeness, we also provide a result concerning negative previsions for events.

THEOREM 3. *Let $\{A_i\}_{i=1}^n$ be a partition with $n > 1$. Let $p_1, \ldots, p_k \geq 0$ be lower previsions for $A_1, \ldots, A_k$ $(k < n)$ and let $q_{k+1}, \ldots, q_n < 0$ be upper previsions for $A_{k+1}, \ldots, A_n$. Assume that all A_i are nonempty.*

- *Suppose that $k \geq 1$. Let $p = \sum_{i=1}^k p_i$, and let $r = \max\{p-1, -q_{k+1}, \ldots, -q_n\}$. Then $\rho = r/(1+r)$. If $p - 1 = r$, then a gamble that achieves the minimum in (2.3) has $\alpha_i = 0$ for $i > k$ and $\alpha_i = 1/r$ for $i \leq k$. If $-q_j = r$, then a gamble that achieves the minimum in (2.3) has $\alpha_i = 0$ for all $i \neq j$ and $\alpha_j = 1/(1 - q_j)$.*

- *Suppose that $k = 0$ (that, is all events have negative upper prevision). Then $\rho = [1 - \sum_{i=1}^n q_i]/\sum_{i=1}^n (1 - q_i)$, and $\alpha_i = 1/\sum_{i=1}^n (1 - q_i)$ for all i achieves this value.*

In the next section, we show how Theorem 2 applies to the example of hypothesis testing at a fixed level.

3. Hypothesis testing as an example. Lindley (1972, p. 14) argues that it is incoherent to test all hypotheses at the same level, such as .05. (See also Seidenfeld, Schervish and Kadane, 1990.) Cox (1958) gave an example of how testing all hypotheses at the same level leads to inadmissibility. In this section, we show how this incoherence and inadmissibility can be measured using the measure of incoherence ρ.

Consider the case of testing a simple hypothesis against a simple alternative. Let f_0 and f_1 be two possible densities for a random quantity X, and let f be the "true" density of X. Suppose that we wish to test the hypothesis $H : f = f_0$ versus the alternative $A : f = f_1$. To write this as a decision problem, let the action space be $\{0, 1\}$ where action $a = 0$ corresponds to accepting H and action $a = 1$ corresponds to rejecting H. Let the loss function have the form

$$L(f, a) = \begin{cases} c_1 & \text{if } f = f_1 \text{ and } a = 0, \\ c_0 & \text{if } f = f_0 \text{ and } a = 1, \\ 0 & \text{otherwise,} \end{cases}$$

with $c_0, c_1 > 0$. The Neyman-Pearson lemma says that the most powerful tests of their sizes and the Bayes rules with respect to all priors have the form: For some constant k, choose $a = 1$ if $f_1(x) > kf_0(x)$, choose $a = 0$ if $f_1(x) < kf_0(x)$ and do whatever you wish (even randomization) if $f_1(x) = kf_0(x)$. Now, suppose that someone chooses a value of k and declares that they prefer the corresponding test to all other tests. One could infer from this choice an "implied prior" over the two possibilities f_0 and f_1. If $\Pr(f = f_0) = p$ and $\Pr(f = f_1) = 1-p$, then the Bayes rule is to choose $a = 1$ if $pc_0f_0(x) < (1-p)c_1f_1(x)$, which corresponds to $k = pc_0/[(1-p)c_1]$. So $p = c_1k/(c_0 + c_1k)$.

Of course, a classical statistician who refuses to use prior and posterior probabilities will not acknowledge the implied prior. However, incoherence will arise if two tests about the same parameter imply different priors. We illustrate this with a version of the example of Cox (1958). Let $c_0 = c_1 = 1$ and let f_0 and f_1 be normal distributions with different means θ but the same variance σ^2. Suppose that the hypothesis is $H : \theta = 0$ versus $A : \theta = 1$. (The phenomenon we illustrate here applies more generally but the greater generality does not contribute to understanding.) Suppose that either $\sigma = 1$ or $\sigma = 0.3$ will be true, but we will not know which until we observe the data. That is, the data consist of the pair (X, σ). Let $\Pr(\sigma = 1) = 0.5$, so that σ is ancillary. A classical statistician who prefers level 0.05 tests whenever available might think that, after observing σ a conditional level 0.05 test should still be preferred to a test whose conditional level given σ is something else. The most powerful conditional level 0.05 test is to reject $H : \theta = 0$ if $X > 1.645\sigma$. The most powerful marginal level 0.05 test rejects H if $X > 0.5 + 0.9438\sigma^2$ and is the Bayes rule with respect to the prior $\Pr(\theta = 0) = 0.7199$. The marginal power of the Bayes rule is 0.6227, while the marginal power of the conditional level 0.05 test is 0.6069. Since both tests have the same level, the conditional test is inadmissible.

Before computing our measure of incoherence, we consider a slightly more general setting in which the two possible values of σ are arbitrary and the distribution of σ is arbitrary. Suppose that either $\sigma = \sigma_1$ or $\sigma = \sigma_2$ and let $r_i = \Pr(\sigma = \sigma_i)$. The implied prior probability of $\theta = 0$ is $k/(1 + k)$ where k comes from the form of the Neyman-Pearson test. If a classical statistician decides to test the hypothesis $H : \theta = 0$ at level .05 regardless of the value of σ, this will often require choosing two different values of k in the Neyman-Pearson lemma. In fact, $k = \exp(1.645/\sigma - 0.5/\sigma^2)$, and the implied prior is

$$p(\sigma) = \left[1 + \exp\left(\frac{-1.645}{\sigma} + \frac{0.5}{\sigma^2}\right)\right]^{-1}, \tag{3.1}$$

not a one-to-one function of σ, but highly dependent on σ.

Given the value of σ, the level .05 test is

$$\delta_{cl}(x, \sigma) = \begin{cases} 1 & \text{if } x > 1.645\sigma, \\ 0 & \text{if } x \leq 1.645\sigma, \end{cases}$$

where 1 means to reject H and 0 means to accept H. The risk function of δ_{cl} (given σ) for the classical statistician is

$$R(\theta, \delta_{cl}|\sigma) = \begin{cases} .05 & \text{if } \theta = 0, \\ 1 - \beta_{cl}(\sigma) & \text{if } \theta = 1, \end{cases} \tag{3.2}$$

where

$$\beta_{cl}(\sigma) = \Phi\left(\frac{1}{\sigma} - 1.645\right)$$

is the power of the level .05 test, and Φ is the standard normal distribution function. For each σ, the classical statistician prefers this risk function to the risk function of every other test. Let δ be some other test whose size is $\alpha(\sigma)$ and whose power is $\beta(\sigma)$. Assume that δ_{cl} does not dominate δ since a dominated test would not play any role in illustrating the incoherence of δ_{cl}. In particular, this implies that $\alpha(\sigma) \neq .05$, since another size .05 test would be dominated by δ_{cl}. The risk function of the test δ is

$$R(\theta, \delta|\sigma) = \begin{cases} \alpha(\sigma) & \text{if } \theta = 0, \\ 1 - \beta(\sigma) & \text{if } \theta = 1. \end{cases}$$

Since the classical statistician prefers to lose the risk $R(\theta, \delta_{cl}|\sigma)$ to losing $R(\theta, \delta|\sigma)$ we should be allowed to offer a trade in which the gain to the classical statistician is the difference between these two risk functions. That is, create the gamble

$$\begin{aligned} R(\theta, \delta|\sigma) - R(\theta, \delta_{cl}|\sigma) &= \begin{cases} \alpha(\sigma) - .05 & \text{if } \theta = 0, \\ \beta_{cl}(\sigma) - \beta(\sigma) & \text{if } \theta = 1 \end{cases} \\ &= a(I_A - b), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A &= \{\theta = 0\}, \\ a &= \alpha(\sigma) - .05 + \beta(\sigma) - \beta_{cl}(\sigma), \\ b &= \frac{\beta(\sigma) - \beta_{cl}(\sigma)}{\alpha(\sigma) - .05 + \beta(\sigma) - \beta_{cl}(\sigma)}. \end{aligned}$$

The gamble $a(I_A - b)$ is acceptable, even favorable, to the classical statistician. In other words, b is an upper or lower prevision for A depending on whether $a < 0$ or $a > 0$. Since neither δ_{cl} nor δ dominate each other, $\alpha(\sigma) - .05$ and $\beta(\sigma) - \beta_{cl}(\sigma)$ have the same sign. So, b is an upper prevision for A if $\alpha(\sigma) < .05$ and b is a lower prevision for A if $\alpha(\sigma) > .05$.

We can repeat all of this for each of the two possible values of σ, obtaining two tests δ_1 and δ_2 whose risk functions we can trade for those of $\delta_{cl}(\sigma_1)$ and $\delta_{cl}(\sigma_2)$ respectively. This produces two gambles of the form (3.3), call them $a_1(I_A - b_1)$ and $a_2(I_A - b_2)$. It is easy to see that, in order for a combination of two such gambles with positive coefficients to be uniformly negative, it is necessary that a_1 and a_2 have opposite signs. This means that one of b_1 and b_2 is an upper prevision and the other is a lower prevision. There is incoherence if and only if the lower prevision is greater than the upper prevision. Let b_2 be the upper prevision. We can then write $a_2(I_A - b_2) = -a_2(I_{A^C} - [1 - b_2])$. This allows us to treat $p_1 = b_1$ and $p_2 = 1 - b_2$ as lower previsions for the elements of a partition $\{A, A^C\}$. There is incoherence if and only if the lower previsions add up to more than 1.

Assuming that $p_1 + p_2 > 1$, Theorem 2 says that the maximum rate of guaranteed loss (for fixed tests δ_1 and δ_2) is $\rho = (s - 1)/s$ where $s = p_1 + p_2$. This is an increasing function of $b_1 - b_2$. If we can choose δ_1 and δ_2 so as to make $b_1 - b_2$ as large as possible, we will be able to make ρ as large as possible. We can write $b_i = v_i/(1 + v_i)$ for $i = 1, 2$, where

$$v_i = \frac{\beta(\sigma_i) - \beta_{cl}(\sigma_i)}{\alpha(\sigma_i) - .05}.$$

Notice that b_i depends only on δ_i and σ_i for each i. Hence, we can maximize $b_1 - b_2$ by maximizing b_1 and simultaneously minimizing b_2. Equivalently, we can maximize v_1

and minimize v_2. Since we assumed that $\alpha(\sigma_1) > .05$, we make v_1 as large as possible by making sure that δ_1 has the largest power among all tests of its size. Similarly, since $\alpha(\sigma_2) < .05$, we make v_2 as small as possible by making sure that δ_2 has the largest power among all tests of its size. In other words, δ_1 and δ_2 should be Bayes rules with respect to some priors P_1, P_2 on $\{0,1\}$. Let $\gamma_i = \log P_i(0)/P_i(1)$. Let $\alpha_\gamma(\sigma)$ and $\beta_\gamma(\sigma)$ stand for the size and power respectively of the Bayes rule with respect to the prior that satisfies $\gamma = \log \Pr(\theta = 0)/\Pr(\theta = 1)$. Then $v_i = v(\gamma_i, \sigma_i)$ where

$$v(\gamma, \sigma) = \frac{\beta_\gamma(\sigma_i) - \beta_{cl}(\sigma_i)}{\alpha_\gamma(\sigma_i) - .05}.$$

Let $\lambda_i = \log p(\sigma_i)/[1 - p(\sigma_i)]$, where $p(\sigma_i)$ is the implied prior from (3.1). It follows from $\alpha(\sigma_2) < .05 < \alpha(\sigma_1)$ that $\lambda_1 > \gamma_1$ and $\lambda_2 < \gamma_2$.

It is easy to see that $-v_i$ is the slope of the line that connects the two points $(\alpha_\gamma(\sigma_i), 1-\beta_\gamma(\sigma_i))$ and $(.05, 1-\beta_{cl}(\sigma_i))$ which are on the lower boundary of the risk set for this decision problem. (For more detail on risk sets, see Sections 3.2.4 and 4.3.1 of Schervish, 1995.) Since $\gamma = -\infty$ corresponds to the point $(1,0)$ and $\gamma = \infty$ corresponds to $(0,1)$ and since the risk set is convex, it follows from the mean value theorem that $-v(\gamma, \sigma)$ is decreasing as a function of γ for fixed σ. This makes b_1 an increasing function of γ_1 and makes b_2 is an increasing function of γ_2. To maximize $b_1 - b_2$, we need γ_1 to be as large as possible subject to $\gamma_1 < \lambda_1$ and we need γ_2 to be as small as possible subject to $\gamma_2 > \lambda_2$. Of course, if we were to set $\gamma_1 = \lambda_1$ and $\gamma_2 = \lambda_2$, we would reproduce the conditional level .05 test. But we can get arbitrarily close to the maximum possible by letting γ_1 and γ_2 get arbitrarily close to λ_1 and λ_2. The limit of b_1 as γ_1 approaches λ_1 can be found by L'Hôpital's rule (or from the strict convexity of the lower boundary of the risk set) to equal $p(\sigma_1)$ from (3.1), the implied prior corresponding to σ_1. Similarly, the limit of q_2 is $p(\sigma_2)$.

In summary, we have shown that, by choosing γ_1 arbitrarily close to (but smaller than) $1.645/\sigma_1 - 0.5/\sigma_1^2$ and by choosing γ_2 arbitrarily close to (but larger than) $1.645/\sigma_2 - 0.5/\sigma_2^2$ we can make ρ arbitrarily close to $[p(\sigma_1) - p(\sigma_2)]/[1 + p(\sigma_1) - p(\sigma_2)]$. This last value is exactly what would be obtained from Theorem 2 if a bookie had offered lower previsions of $p(\sigma_1)$ and $1 - p(\sigma_2)$ for $\{\theta = 0\}$ and $\{\theta = 1\}$. A similar argument would arrive at the same result if we had interpreted $1 - p(\sigma_1)$ and $p(\sigma_2)$ as incoherent upper previsions.

4. The gambler's perspective.

Suppose next that we require the gambler instead of the bookie to escrow funds to cover his/her bets. This might correspond to an investor who notices several options being offered by a single bookie or several different bookies. This investor might be asked to exhibit sufficient unencumbered funds to cover the maximum gain to the bookie for each investment. Using the same notation as before, a gamble $\alpha(X - x)$ pays the bookie (the investor or gambler loses) if $\alpha(X - x) > 0$. So the amount of escrow for this gamble should be $\sup_{t \in T} \alpha(X(t) - x)$. For a combination $Y = \sum_{i=1}^n \alpha_i(X_i - y_i)$ of acceptable gambles, define the *guaranteed profit* to the gambler as the guaranteed loss to the bookie, namely $G(Y) = -\min\{0, \sup_{t \in T} Y(t)\}$. Define ψ, the *maximum rate of guaranteed profit*, as the supremum, over all combinations Y of acceptable gambles, of $G(Y)$ subject to the gambler's escrow for Y being at most 1. The maximum rate of guaranteed profit is defined just like the maximum rate of guaranteed loss but with a different normalization. As in Theorem 1, we can prove that if the previsions are incoherent, then

$$\psi = -\inf_{\alpha_1,\ldots,\alpha_n} \frac{g(\alpha_1,\ldots,\alpha_n)}{h'(\alpha_1,\ldots,\alpha_n)}, \tag{4.1}$$

where $h'(\alpha_1,\ldots,\alpha_n) = \sum_{i=1}^n \max\{0, \sup_{t\in T}(X_i(t) - x_i)\}$. That is, when calculating ψ, it does not matter whether or not $\alpha(X - x)$ is acceptable when x is an upper or lower prevision for X. The difference between rate of loss and rate of profit is more than cosmetic, however.

EXAMPLE 5. Let A be an event with lower prevision $p > 1$, then for every $\alpha > 0$, $\sup_{t\in T} \alpha(I_A(t) - p) < 0$ and $h'(\alpha) = 0$. It follows easily that $\psi = \infty$. This differs dramatically from the case in which we standardize on the incoherent bookie's escrow. In that case $\inf_{t\in T} \alpha(I_A(t) - p) = -\alpha p$ and $\rho = [p-1]/p$.

EXAMPLE 6. Let $A_2 = A_1^C$ with lower previsions $0 < p_i < 1$ for $i = 1, 2$ but $p_1 + p_2 = q > 1$, then for $\alpha_1, \alpha_2 > 0$ $h'(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2 - \alpha_1 p_1 - \alpha_2 p_2$. If $h' = 1$, then $g(\alpha_1, \alpha_2) = 1 - \min\{\alpha_1, \alpha_2\}$. It is not difficult to show that minimizing g subject to $h' = 1$ leads to $\alpha_1 = \alpha_2 = 1/[2-q]$ and $\psi = [q-1]/[2-q]$. (This is a special case of Theorem 4 below.) This is similar to the bookie's perspective in which $\rho = [q-1]/q$ and $\alpha_1 = \alpha_2 = 1/q$ provides the maximum guaranteed loss. One should note that $2 - q < q$ in this case, so the gambler wins more per unit escrow than the bookie loses. Put another way, the gambler has to escrow less per unit of winnings than the bookie escrows per unit of loss. So we see that there are cases in which the two perspectives are just equivalent but distinct normalizations and there are cases in which they are very different.

The differences between the bookie's and gambler's perspectives are sometimes more subtle than in Example 5.

EXAMPLE 7. Let A_1, A_2, A_3 be a partition with lower previsions $p_1 = 0.6$, $p_2 = 0.7$ and $p_3 = 0.7$. If the gambler takes bets with all three α_i equal, then $\alpha_i = 1$ and the guaranteed profit is $1/1 = 1$. If $p_3 = 0.2$ instead, and the gambler takes bets with all three α_i equal, then $\alpha_i = 1/1.5$ and the guaranteed profit is $0.5/1.5 = 0.3333$. In this latter case, however, it is better for the gambler to only take bets with $\alpha_3 = 0$. With $\alpha_1 = \alpha_2 = 1/0.7$, the guaranteed profit is $0.3/0.7 = 0.4286$, which is higher than the gambler can get with all three α_i equal. The reasons for the difference in behavior in these two examples are given in Theorem 4 below and the ensuing discussion.

There is an analog to Result 1 from the gambler's perspective.

RESULT 2. *Let $A_1, \ldots, A_m$ be disjoint events with upper previsions $q_1, \ldots, q_m \geq 0$. Let $B = (\cup_{i=1}^m A_i)^C$. If $\sum_{i=1}^n q_i \leq 1$, then, the value of ψ is unchanged if we can act as if $p = 1 - \sum_{i=1}^m q_i$ were a lower prevision for B.*

The proof of Result 2 is so similar to the proof of Result 1 that we omit it.

Due to the availability of infinite guaranteed profit when lower previsions for events are greater than 1 or when upper previsions are less than 0, we assume in this section that every upper prevision and every lower prevision for an event is between 0 and 1.

THEOREM 4. *Let $\{A_i\}_{i=1}^n$ be a partition with $n > 1$ and all A_i nonempty. Assume that $n > 1$ and that all A_i are nonempty. Let $0 \le q_1, \ldots, q_n \le 1$ be upper previsions, and let $0 \le p_1 \le \ldots \le p_n \le 1$ be lower previsions. Define $A_1, \ldots, A_n$ and let $s^+ = \sum_{i=1}^n q_i$ and $s^- = \sum_{i=1}^n p_i$.*

1. *If $s^+ < 1$, then $\alpha_j = -1/s^+$ for all j achieves the* inf *in (4.1) and $\psi = [1 - s^+]/s^+$.*
2. *If $s^- > 1$, let k^* be the first k such that $\sum_{i=n-k+1}^n p_i > 1 + (k-1)p_{n-k}$ ($k^* = n$ if the inequality is never satisfied). Then $\alpha_j = 1/\sum_{i=n-k^*+1}^n (1 - p_i)$ for all $j \ge n - k^* + 1$ and $\alpha_j = 0$ for $j < n - k^*$ achieves the* inf *in (4.1) and $\psi = [\sum_{i=n-k^*+1}^n p_i - 1]/\sum_{i=n-k^*+1}^n (1 - p_i)$.*

As before, it is impossible for both $s^+ < 1$ and $s^- > 1$.

EXAMPLE 8. In Example 7, in the case with $p_3 = 0.7$, we have $k^* = 3$ since $0.7 + 0.7 < 1 + (2-1)0.6$. In the case with $p_3 = 0.2$ we have $k^* = 2$ because $0.7 + 0.6 > 1 + (2-1)0.2$. This explains why the gambler's investment strategies are different in these two cases. If $p_3 = 0.3$, then both strategies ($\alpha_i = 1/1.4$ all i or $\alpha_1 = \alpha_2 = 1/0.7, \alpha_3 = 0$) lead to the same guaranteed profit, $0.6/1.4 = 0.4286$. This is one of the reasons that Theorem 4 is stated without any uniqueness claims. Even in this nonunique case, the gambler would presumably prefer the combination with $\alpha_1 = \alpha_2 = 1/0.7$, because the profit when A_3 occurs will be 1.857, much higher than the guaranteed profit (obtained when $A_1 \cup A_2$ occurs).

The differences between the betting strategies in Examples 7 and 8 is understandable in the following terms. When $p_3 = 0.2$, the existing incoherence exhibited by p_1 and p_2 is sufficiently large that it does not make sense for the gambler to waste any resources (escrow) on the gamble $\alpha(I_{A_3} - 0.2)$ that will only add a little bit to the gross profit while taking away from the amount that can be invested in the larger incoherence involving $A_1 \cup A_2$. When $p_3 = 0.7$, however, there is enough additional gain available to the gambler by allowing the gamble $\alpha(I_{A_3} - 0.7)$ to make it profitable to invest some in this gamble and take a little away from the others. We illustrate this phenomenon geometrically in Figure 1. The four planes shown are where $\sum_{i=1}^3 p_i = 0.5, 1, 1.5, 2$ respectively. The solid-shaded areas of the incoherent planes are the regions where the gambler combines all three gambles. The line-shaded areas are where the gambler combines only two gambles (not betting on the event with the smallest probability). Note that the solid-shaded regions are obtained as follows. Form a triangular cone with vertex $(1,1,1)$ and three edges (broken lines in Figure 1) passing through the corners of the coherent simplex ($\sum_{i=1}^3 p_i = 1$). The intersections of this cone with each of the incoherent planes $\sum_{i=1}^3 p_i = c$ are the solid-shaded regions. Where the cone extends below the coherent plane, it encloses all of the incoherent planes $\sum_{i=1}^3 p_i = c < 1$, thereby illustrating the first part of Theorem 4.

5. Simple random variables. Consider next a partition $\{A_i\}_{i=1}^n$ of T and a simple random variable $X = \sum_{i=1}^n x_i I_{A_i}$, where the x_i are distinct. We assume that all events and random quantities have previsions assessed rather than just upper and/or lower previsions. Incoherence can be exhibited in several ways. One way would be if the prevision of X were not in the interval $[x_{(1)}, x_{(n)}]$, where $x_{(1)} = \min\{x_1, \ldots, x_n\}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\}$. Another way would be if the bookie had assigned incoherent previsions $p_i = P(A_i)$ for

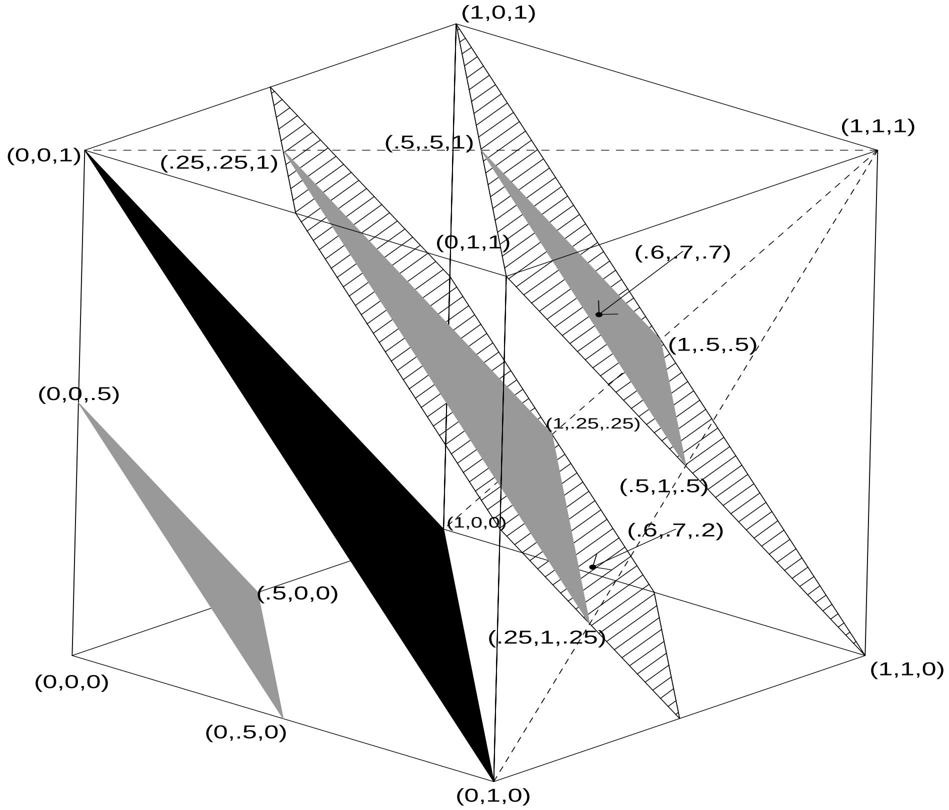

FIG. 1. *Sets of bookie's previsions for which the gambler uses all three events (grey shading) or just two events (line shading) in Example 8. The coherent plane is shaded black. The broken lines give the edges of the triangular cone.*

$i = 1, \ldots, n$ such that $s = \sum_{i=1}^n p_i \neq 1$ to the indicators of the partition elements. Yet another way would be if the prevision p_X of X were not equal to $\mu = \sum_{i=1}^n x_i p_i$. In this last case,

$$\alpha(X - p_X) - \sum_{i=1}^{n} (\alpha x_i + c)(I_{A_i} - p_i) = \alpha(\mu - p_X) + c(s - 1) \tag{5.1}$$

in all circumstances. So, let $\alpha > 0$ if $\mu < p_X$ and $\alpha < 0$ if $\mu > p_X$. Also, let $c < 0$ if $s > 1$ and $c > 0$ if $s < 1$. This provides a constant loss for the bookie (profit to the gambler) no matter what happens. The reason that we consider only previsions in this section rather than upper and lower previsions is that we do not have sufficient control of the signs of the coefficients in (5.1). For example, suppose that $s > 1$ and that the p_i are lower previsions. If $\mu < p_X$, we want $\alpha > 0$ in which case $-\alpha x_i - c$ will be positive only if $c < -x_i\alpha$ for all i. If the x_i are positive, this might require a large escrow. On the other hand, if the p_i are previsions, we are not required to have $-\alpha x_i - c > 0$ for all i and it becomes easier to maximize the rate of loss or rate of profit. Indeed, examples are given in which values of $-\alpha x_i - c$ for different i need to have opposite signs in order to maximize the rate of loss.

5.1. *The bookie's perspective.* Theorem 5 proves that a version of (5.1) provides the maximum rate of guaranteed loss.

THEOREM 5. *Let $X = \sum_{i=1}^n x_i I_{A_i}$, where $\{A_i\}_{i=1}^n$ forms a partition of T and the x_i are distinct. Assume that $n > 1$ and all A_i are nonempty. Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be* *the ordered values of* $x_1, \ldots, x_n$. *Let* $p_i = P(A_i)$ *and* $p_X = P(X)$ *be previsions such that* $x_{(1)} \leq p_X \leq x_{(n)}$ *and* $0 \leq p_i \leq 1$ *for all* i. *Let* $\mu = \sum_{i=1}^n x_i p_i$, $\delta = p_X - \mu$ *and* $s = \sum_{i=1}^n p_i$. *Define*

$$\begin{aligned} r_-(c) &= \frac{-\delta + c(1-s)}{-\delta + x_{(n)} - cs - \sum_{i=1}^n \min\{x_i - c, 0\}}, \\ r_+(c) &= \frac{\delta + c(1-s)}{\delta - x_{(1)} - cs - \sum_{i=1}^n \min\{-x_i - c, 0\}}. \end{aligned}$$

Then, amongst combinations of gambles of the form

$$\alpha_X(X - p_X) + \sum_{i=1}^n \alpha_i (I_{A_i} - p_i), \tag{5.2}$$

the maximum rate of guaranteed loss is determined as follows:

1. *Suppose that* $s < 1$.
   (a) *If* $\delta/(1-s) < \sum_{i=1}^{n-1} x_{(i)}/(n-1)$ *then* $\rho = \sup_c r_-(c)$. *The rate can be achieved with* $\alpha_X = -1$ *and* $\alpha_i = x_i - c$, *where* c *provides the maximum value of* r_-.
   (b) *If* $\delta/(1-s) > \sum_{i=2}^{n} x_{(i)}/(n-1)$ *then* $\rho = \sup_c r_+(c)$. *The rate can be achieved with* $\alpha_X = 1$ *and* $\alpha_i = -x_i - c$, *where* c *provides the maximum value of* r_+.
   (c) *If* $\sum_{i=1}^{n-1} x_{(i)}/(n-1) \leq \delta/(1-s) \leq \sum_{i=2}^{n} x_{(i)}/(n-1)$, *then then* $\alpha_X = 0$ *and Theorem 2 gives the values of* ρ *and the* α_i.

2. *Suppose that* $s > 1$.
   (a) *If* $\delta/(s-1) < -x_{(n)}$ *then* $\rho = [-\delta - (s-1)x_{(1)}]/[-\delta + x_{(n)} - sx_{(1)}]$. *The rate can be achieved with* $\alpha_X = -1$ *and* $\alpha_i = x_i - x_{(1)}$ *for* $i = 1, \ldots, n$.
   (b) *If* $\delta/(s-1) > -x_{(1)}$, *then* $\rho = [\delta + (s-1)x_{(n)}]/[\delta - x_{(1)} + sx_{(n)}]$. *The rate can be achieved with* $\alpha_X = 1$ *and* $\alpha_i = x_{(n)} - x_i$ *for* $i = 1, \ldots, n$.
   (c) *If* $-x_{(n)} \leq \delta/(s-1) \leq -x_{(1)}$, *then* $\alpha_X = 0$ *and Theorem 2 gives the values of* ρ *and the* α_i.

3. *Suppose that* $s = 1$.
   (a) *If* $\delta < 0$, *then* $\rho = -\delta/(-\delta + x_{(n)} - x_{(1)})$. *The rate can be achieved with* $\alpha_X = -1$ *and* $\alpha_i = x_i - c$ *for* $i = 1, \ldots, n$ *where* c *is an arbitrary number in the interval* $[x_{(1)}, x_{(2)}]$.
   (b) *If* $\delta > 0$, *then* $\rho = \delta/(\delta + x_{(n)} - x_{(1)})$. *The rate can be achieved with* $\alpha_X = 1$ *and* $\alpha_i = c - x_i$ *for* $i = 1, \ldots, n$ *where* c *is an arbitrary number in the interval* $[x_{(n-1)}, x_{(n)}]$.
   (c) *If* $\delta = 0$, *the previsions are coherent.*

Note that in Theorem 5, as in Theorem 6 below, at least one $\alpha_i = 0$ in the combination of gambles that leads to the maximum rate. The reason for this is intuitively simple. A combination such as (5.2) has the property (when $s = 1$) that it produces the same value in every state if the α_i are all replaced by $\alpha_i + v$ for arbitrary constant v. However, the escrow changes in a more complicated way. In fact, escrows for gambles of the form $\alpha_i(I_{A_i} - p_i)$ with $p_i \leq 1/2$ tend to be larger for $\alpha_i < 0$ than for $\alpha_i > 0$. With $s = 1$, at most one p_i can be greater than 1/2. So, Theorem 5 chooses the value v to make all of the $\alpha_i \geq 0$ and one of them exactly 0 so that the one gamble will need no escrow at all.

Theorem 5 does not give a closed-form result for cases in which $s = \sum_{i=1}^n p_i < 1$. However, in the proof, we show that the functions r_- and r_+ can change from increasing to decreasing only at one of the values $\pm x_1, \ldots, \pm x_n$. Except in rare circumstances, the maximum is unique when $s \neq 1$ because the derivative of the ratio will not be 0. It is possible, with a fortuitous combination of values, for the ratio to be flat over the interval between two x values. For this reason, there is no uniqueness claim in Theorem 5. In fact, when $s = 1$, there is always an interval of c values that leads to the same rate of sure loss.

EXAMPLE 9. Let A_1, A_2, A_3 be a partition and let $X = I_{A_2} + 2I_{A_3}$. Suppose that $p_1 = 0.1$, $p_2 = 0.5$ and $p_3 = 0.1$, where $p_i = P(A_i)$. Then $s = 0.7 < 1$ and $\mu = 0.7$. The bookie needs to choose a value for p_X. For each possible value of p_X Theorem 5 tells us how a gambler can maximize the rate of guaranteed loss by choosing the coefficients for the various gambles. We summarize the results in the following table. The intervals in the

TABLE 1
Coefficients to maximize ρ in Example 9

p_X	α_X	α_1	α_2	α_3	ρ
$[0, 0.4)$	-1	-1	0	1	[0.3333,0.2288)
$[0.4, 0.85)$	-1	-2	-1	0	[0.2288,0.1304)
$[0.85, 1.15]$	0	-1	-1	-1	0.1304
(1.15,1.6)	1	0	-1	-2	(0.1304,0.2308)
$[1.6, 2]$	1	1	0	-1	[0.2308,0.3333]

left column give the ranges of p_X values for which the coefficients of the various gambles all have the same values. The intervals in the right column give the ranges of ρ values that are obtained for the corresponding values of p_X. For example, as p_X increases from 0 to 0.4, ρ decreases from 0.3333 to 0.2288. If p_X equals one of the values 0.4, 0.85, 1.15 or 1.6, then there is a large collection of ways to achieve the maximum rate of guaranteed loss corresponding to an interval of c values in Theorem 5.

EXAMPLE 10. Let A_1, A_2, A_3 be a partition and let $X = I_{A_2} + 2I_{A_3}$. Suppose that $p_1 = 0.2$, $p_2 = 0.7$ and $p_3 = 0.2$, where $p_i = P(A_i)$. Then $s = 1.1$ and $\mu = 1.1$. The bookie needs to choose a value for p_X. For each possible value Theorem 5 tells how a gambler can maximize the rate of guaranteed loss by choosing the coefficients for the various gambles. We summarize the results in the following table. The intervals in the left column give the ranges of p_X values for which the coefficients of the various gambles all have the same values. The intervals in the right column give the ranges of ρ values that are obtained for the corresponding values of p_X. For example, as p_X increases from 0 to 0.9, ρ decreases from 0.3548 to 0.0909. If p_X equals 0.9 or 1.1, there will be a large collection of ways to achieve the maximum rate of guaranteed loss corresponding to an interval of c values in Theorem 5.

TABLE 2
Coefficients to maximize ρ in Example 10

p_X	α_X	α_1	α_2	α_3	ρ
$[0, 0.9)$	-1	0	1	2	[0.3548,0.0909)
$[0.9, 1.1]$	0	1	1	1	0.0909
(1.1,2]	1	2	1	0	(0.0909,0.3548]

A simple corollary is available for Theorem 5. Suppose that the bookie has already announced $P(A_i)$ for $i = 1, \ldots, n$ and now is trying to choose $P(X)$, where $X = \sum_{i=1}^{n} x_i I_{A_i}$. Theorem 5 gives bounds on $P(X)$ that allow the bookie to not increase the value of ρ. Specifically, the cases labeled (c) must occur.

COROLLARY 1. *Let $X = \sum_{i=1}^{n} x_i I_{A_i}$, where $\{A_i\}_{i=1}^{n}$ forms a partition of T and the x_i are distinct. Assume that $n > 1$ and all A_i are nonempty. Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be the ordered values of $x_1, \ldots, x_n$. Let $p_i = P(A_i)$ for $i = 1, \ldots, n$ and $\mu = \sum_{i=1}^{n} p_i x_i$. The addition of the prevision $p_X = P(X)$ will not increase ρ (the maximum rate of guaranteed loss) if and only if it is chosen as follows:*

$$\begin{aligned} \mu + \frac{1-s}{n-1}\sum_{i=1}^{n-1} x_{(i)} \le p_X \le \mu + \frac{1-s}{n-1}\sum_{i=2}^{n} x_{(i)} \qquad & \textit{if } s < 1, \\ \max\{x_{(1)}, \mu - (s-1)x_{(n)}\} \le p_X \le \min\{x_{(n)}, \mu - (s-1)x_{(1)}\} \qquad & \textit{if } s > 1, \\ p_X = \mu \qquad & \textit{if } s = 1. \end{aligned}$$

Notice how the range of allowable p_X values widens as s moves away from 1 in both the $s > 1$ and $s < 1$ cases in Corollary 1. This is due to the fact that, the farther s is from 1, the easier it is to create large losses without betting on X at all. In order to justify placing bets on X, its prevision will have to be all the more extreme. Also, notice that choosing $p_X = \mu$ does not always prevent ρ from increasing. In Example 9, we had $\mu = 0.7$, but $p_X = 0.7$ still makes ρ larger (0.1667) than it was based solely on the incoherent previsions of the three partition elements (0.1304). However,

$$p_X = \mu + (1-s)\sum_{i=2}^{n-1} x_{(i)}/(n-2) \tag{5.3}$$

prevents ρ from increasing if $n > 2$.

5.2. *The gambler's perspective.* Recall that Theorem 4 provides a rather complicated result from the gambler's perspective when the sum of the previsions of partition elements is greater than 1. For this reason, we will consider the analog of Theorem 5 only in the case $s \le 1$. Indeed, examples suggest that the general result for $s > 1$ will be very complicated to state, let alone prove. In this section, it is convenient to assume, without loss of generality, that the values $x_1, \ldots, x_n$ are ordered from smallest to largest.

THEOREM 6. *Let $X = \sum_{i=1}^{n} x_i I_{A_i}$, where $\{A_i\}_{i=1}^{n}$ forms a partition of T and $x_1 < x_2 < \cdots < x_n$. Let $n > 1$ and assume that all A_i are nonempty. Let $p_i = P(A_i)$*

and $p_X = P(X)$ be previsions such that $x_1 \le p_X \le x_n$ and $0 \le p_i \le 1$ for all i. Let $s = \sum_{i=1}^n p_i \le 1$. Let $\mu = \sum_{i=1}^n p_i x_i$ and $\delta = p_X - \mu$. For each $k = 1, \ldots, n$, define

$$s_{-,k} = \sum_{i=1}^{k} p_i, \qquad v_-(k) = p_X - \sum_{i=1}^{k} x_i p_i - (1 - s_{-,k})x_1,$$

$$s_{+,k} = \sum_{i=n-k+1}^{n} p_i, \qquad v_+(k) = (1 - s_{+,k})x_n + \sum_{i=n-k+1}^{n} x_i p_i - p_X.$$

Then, amongst combinations of gambles of the form

$$\alpha_X(X - p_X) + \sum_{i=1}^{n} \alpha_i (I_{A_i} - p_i), \tag{5.4}$$

the maximum rate of guaranteed profit is determined as follows:

1. *If $\delta < (1-s)x_1$, let k^* be the smallest value of k such that $v_-(k) < 0$. Then set $\alpha_X = -1$, $\alpha_i = x_i - x_{k^*}$ for $i \le k^*$ and $\alpha_i = 0$ for $i > k^*$.*
2. *If $\delta > (1-s)x_n$, let k^* be the smallest value of k such that $v_+(k) < 0$. Then set $\alpha_X = 1$, $\alpha_i = -x_i + x_{n-k^*}$ for $i \ge n - k^*$ and $\alpha_i = 0$ for $i < n - k^*$.*
3. *If $(1-s)x_1 \le \delta \le (1-s)x_n$, then set $\alpha_X = 0$ and apply Theorem 4.*

If $s = 1$ and exactly one $\alpha_i = 0$, the same rate is achieved if α_i is replaced by $\alpha_i + c$ for $0 \le c \le \beta$, where $-\beta$ is the second largest α_i. Also, any larger value of c would make the rate of guaranteed profit go down.

Theorem 6 has a simple corollary in the spirit of Corollary 1.

Corollary 2. *Under the conditions of Theorem 6, the addition of the prevision $p_X = P(X)$ to the previsions $p_i = P(A_i)$ for $i = 1, \ldots, n$ does not increase ψ (the maximum rate of guaranteed profit) if and only if $\mu + (1-s)x_1 \le p_X \le \mu + (1-s)x_n$.*

Corollary 2 says that p_X adds no additional incoherence from the gambler's perspective if and only if it is like a recomputation of μ after distributing the missing probability $1-s$ amongst the the x_i in an arbitrary way. That is, $p_X = \sum_{i=1}^n q_i x_i$ with $\sum_{i=1}^n q_i = 1$ and $q_i \ge p_i$ for all i.

Example 11. Consider the same situation as in Example 9 this time from the gambler's perspective. That is, A_1, A_2, A_3 is a partition and $X = I_{A_2} + 2I_{A_3}$ with $p_1 = 0.1$, $p_2 = 0.5$ and $p_3 = 0.1$, where $p_i = P(A_i)$. Then $s = 0.7 < 1$ and $\mu = 0.7$. For each possible value of p_X Theorem 6 tells us how a gambler can maximize the rate of guaranteed profit by choosing the coefficients for the various gambles. We summarize the results in the following table, similar to Table 1. The intervals in the left column give the ranges of p_X values for which the coefficients of the various gambles all have the same values. The intervals in the right column give the ranges of ψ values that are obtained for the corresponding values of p_X. For example, as p_X increases from 0 to 0.5, ψ decreases from 9.0 to 0.6667. If p_X equals one of the values 0.5, 0.7,1.3 or 1.5, there will be a large collection of ways to achieve the maximum rate of guaranteed profit.

TABLE 3
Coefficients to maximize ψ in Example 11

p_X	α_X	α_1	α_2	α_3	ψ
$[0, 0.5)$	-1	-1	0	0	[9.0,0.6667)
$[0.5, 0.7)$	-1	-2	-1	0	[0.6667,0.4286)
$[0.7, 1.3]$	0	-1	-1	-1	0.4286
(1.3,1.5)	1	0	-1	-2	(0.4286,0.6667)
$[1.5, 2]$	1	0	0	-1	[0.6667,9.0]

Notice that, if $p_X \in [0.5, 0.7] \cup [0.85, 1.15] \cup [1.3, 1.5]$ then the coefficients that maximize ψ in Example 11 are the same as the coefficients that maximize ρ in Example 9. For other values of p_X, different coefficients are required to maximize ρ and ψ. When we considered gambles on events only, we found that there were also situations in which the same coefficients maximized both ρ and ψ and others in which different coefficients were required. However, if $s = 1$ and if a simple random variable takes at least four values, there are no combinations of gambles that simultaneously maximize both ρ and ψ unless $\alpha_X = 0$. Part 3 of Theorem 5 says that at most one α_i will be negative if $\alpha_X \neq 0$ and $s = 1$, and if one is negative, it will be either α_1 or α_n. From the additional claims about $s = 1$ in Theorem 6, we see that at most one α_i can be positive while maximizing ψ, and it can only be α_1 or α_n. If $n \geq 4$, no configuration can meet both of these conditions.

For the $s > 1$ case, one can prove one useful fact. Since the ratios (A.21) and (A.22) are monotone functions of each individual α_j on intervals such that α_j does not change sign and the i values at which the maximum occurs do not change, we know that we can split the set $\{1, \ldots, n\}$ into two sets: those j such that $\alpha_j = 0$ and those j such that the minimum ratio occurs at $i = j$. There might be one j in common between these two sets. In general, however, the behavior of the optimal αs is quite complicated as the following example illustrates.

EXAMPLE 12. Let $n = 5$, $x_i = i$ for $i = 1, \ldots, 5$, $p_X = 2$, $p_4 = \epsilon$ and all other $p_i = 0.3$. For sufficiently small ϵ, it would make sense that $\alpha_4 = 0$ as would be the case if we did not have the random variable X (as in Theorem 4). Indeed, if $\epsilon \leq 0.18$, the maximum rate of guaranteed profit is $\psi = 0.21942$ and it occurs with $\alpha_X = \alpha_1 = -1$, $\alpha_2 = \alpha_4 = 0$, $\alpha_3 = 1$ and $\alpha_5 = 3$. For $\epsilon > 0.18$, we need $\alpha_4 = 2$. If $p_4 = 0.3$ but $p_3 = \epsilon$, then $\alpha_4 = 2$ but $\alpha_3 = 0$ when $\epsilon \leq 0.2$. If $p_1 = \cdots = p_4 = 0.3$ but $p_5 = \epsilon$, then $\alpha_5 = 0$ when $\epsilon \leq 0.2$. When two or more p_i are very small compared to the others, the possibilities for 0 α_is become even more complicated.

6. Called-off gambles. In his famous introduction to coherent prevision, de Finetti (1974, Chapter 4) introduces conditional previsions by means of gambles that are called off when some event fails. Consider two events A and B for which a bookie has declared the previsions $p_A = P(A)$ and $p_{AB} = P(A \cap B)$. Suppose that the bookie also needs a prevision $p_{B|A}$ to make the gambles of the form $\alpha I_A(I_B - p_{B|A})$ fair. These gambles are called-off if A fails to occur. It is well-known that the three previsions p_A, p_{AB} and $p_{B|A}$ are coherent if and only if $p_A p_{B|A} = p_{AB}$. This makes $p_{B|A}$ into the usual conditional probability for B given A. Suppose that the previsions satisfy $0 \leq p_{AB} \leq p_A \leq 1$ and $0 \leq p_{B|A} \leq 1$ but $p_A p_{B|A} \neq p_{AB}$. We can rewrite the called-off gamble as

$$\alpha I_A(I_B - p_{B|A}) = \alpha(X - p_{B|A}),$$

where $X = p_{B|A} I_{A^C} + I_{A \cap B}$ is a simple random variable taking the three values $p_{B|A}$, 1 and 0 on the events A^C, $A \cap B$ and $A \cap B^C$ respectively. We will now suppose that the bookie has actually specified the following additional previsions: $P(A^C) = 1 - p_A$, $P(A \cap B^C) = p_A - p_{AB}$ and $P(X) = p_{B|A}$. (It would be incoherent to assign different values, but we will assume that the bookie is at least this coherent.) We now have the setup of Theorems 5 and 6 with $n = 3$, $s = 1$, $p_X = p_{B|A}$ and the correspondences in Table 4. A simple calculation shows that $\mu = (1 - p_A)p_{B|A} + p_{AB}$, so $\delta = p_A p_{B|A} - p_{AB} > (< 0)$

TABLE 4
Correspondences between called-off gambles and the theorems

i	1	2	3
A_i	A^C	$A \cap B$	$A \cap B^C$
p_i	$1 - p_A$	p_{AB}	$p_A - p_{AB}$
x_i	$p_{B\|A}$	1	0

according as $p_{AB} < (>) p_A p_{B|A}$. Theorem 5 says that the maximum rate of guaranteed loss for combinations of gambles of the form

$$\alpha_X(X - p_{B|A}) + \alpha_1(I_{A^C} - [1 - p_A]) + \alpha_2(I_{A \cap B} - p_{AB}) + \alpha_3(I_{A \cap B^C} - [p_A - p_{AB}])$$

is

$$\rho = \frac{|p_{AB} - p_A p_{B|A}|}{|p_{AB} - p_A p_{B|A}| + 1}.$$

If $p_{AB} < p_A p_{B|A}$, then this rate of loss is achieved by every combination of gambles of the form

$$(X - p_{B|A}) + (c - p_{B|A})(I_{A^C} - [1 - p_A]) + c(I_{A \cap B^C} - [p_A - p_{AB}]) + (c - 1)(I_{A \cap B} - p_{AB}), \tag{6.1}$$

for $p_{B|A} \leq c \leq 1$. If $p_{AB} > p_A p_{B|A}$, the rate is obtained by

$$-(X - p_{B|A}) + (p_{B|A} - c)(I_{A^C} - [1 - p_A]) - c(I_{A \cap B^C} - [p_A - p_{AB}]) + (1 - c)(I_{A \cap B} - p_{AB}), \tag{6.2}$$

for $0 \leq c \leq p_{B|A}$. For all c, both (6.1) and (6.2) can be rewritten as

$$\alpha I_A(I_B - p_{B|A}) + \alpha p_{B|A}(I_A - p_A) - \alpha(I_{A \cap B} - p_{AB}) = \alpha(p_{AB} - p_A p_{B|A}), \tag{6.3}$$

with $\alpha = 1$ (if $p_{AB} < p_A p_{B|A}$) or $\alpha = -1$ (if $p_{AB} > p_A p_{B|A}$). However, the escrows for (6.3) are larger than those for (6.2) and (6.1), so the guaranteed rates of loss are smaller.

In this same situation, in order to apply Theorem 6, note that $k^* = 3$ in either of the two cases 1 or 2. It is then possible to write the maximum rate of guaranteed profit for the gambler as

$$\psi = \frac{|p_{AB} - p_A p_{B|A}|}{1 - |p_{AB} - p_A p_{B|A}|}.$$

If $p_{AB} < p_A p_{B|A}$, then this rate of profit is achieved by every combination of gambles of the form (6.1) for $0 \le c \le p_{B|A}$. Note that only $c = p_{B|A}$ achieves both the maximum rate of guaranteed loss and the maximum rate of guaranteed profit. Similarly, if $p_{AB} > p_A p_{B|A}$ the rate is achieved by all gambles of the form (6.2) for $p_{B|A} \le c \le 1$. Once again, only $c = p_{B|A}$ maximizes both the rate of guaranteed loss and the rate of guaranteed profit.

Incoherence can also occur in the previsions of the partition elements $\{A^C, A\cap B, A\cap B^C\}$ when called-off bets are being considered. For the remainder of this discussion, we will use the alternative names A_1, A_2 and A_3 for the partition elements as in Table 4. The gambles of interest are of the form

$$\alpha_X(X - p_X) + \sum_{i=1}^{3} \alpha_i(I_{A_i} - p_i),$$

where $X = \sum_{i=1}^{3} x_i I_{A_i}$ and the p_i and x_i are as in Table 4. For a specific set of previsions p_i for $i = 1, 2, 3$ and p_X, Theorem 5 gives the maximum rate of guaranteed loss for the bookie. However, suppose that the bookie has specified only p_i for $i = 1, 2, 3$ and is now trying to set p_X. In order to avoid making the maximum rate of guaranteed loss increase, the bookie would need to set p_X in accord with Corollary 1. In the case of called-off gambles, we have that $\mu = p_1 p_X + p_2$. If $s < 1$, Corollary 1 says that in order to avoid increasing ρ,

$$p_1 p_X + p_2 + \frac{1-s}{2} p_X \le p_X \le p_1 p_X + p_2 + \frac{1-s}{2}(1 + p_X).$$

These inequalities can be re-expressed as

$$\frac{p_2}{p_2 + p_3 + \frac{1}{2}(1-s)} \le p_X \le \frac{p_2 + \frac{1}{2}(1-s)}{p_2 + p_3 + \frac{1}{2}(1-s)}. \tag{6.4}$$

Note that the inequalities are satisfied if $p_X = p_2/(p_2 + p_3)$, the usual formula for the conditional probability of A_2 given $A_2 \cup A_3$, regardless of the fact that the previsions are incoherent. Similarly, if $s > 1$, to avoid increasing ρ, we need $0 \le p_X \le 1$ and

$$p_1 p_X + p_2 - (s-1) \le p_X \le p_1 p_X + p_2,$$

which can be rewritten

$$\max\left\{0, \frac{p_2 - (s-1)}{p_2 + p_3 - (s-1)}\right\} \le p_X \le \min\left\{1, \frac{p_2}{p_2 + p_3 - (s-1)}\right\}.$$

Once again, $p_X = p_2/(p_2 + p_3)$ always satisfies these inequalities. Note that this value of p_X satisfies (5.3), but there are other values of p_X that avoid increasing ρ.

From the gambler's perspective, with $s < 1$, Corollary 2 says that the bookie avoids increasing ψ when choosing p_X if and only if

$$p_1 p_X + p_2 \le p_X \le p_1 p_X + p_2 + 1 - s,$$

which can be rewritten

$$\frac{p_2}{1-p_1} \leq p_X \leq \frac{p_2+1-s}{1-p_1}. \tag{6.5}$$

Since $p_2+p_3 < 1-p_1$, it is easy to check that $p_X = p_2/(p_2+p_3)$ satisfies these inequalities. In fact, the range (6.5) strictly contains the range (6.4).

7. An alternative definition of coherence. de Finetti (1974) offers a second definition of prevision and coherence which he immediately proves is equivalent to the avoidance of Dutch Book. By this second definition, x is the prevision of X if for all $a > 0$ and $y \neq x$, you prefer to lose $a(X-x)$ to losing $a(X-y)$. Then, a collection of previsions $\{P(X) : X \in \mathcal{X}\}$ for a set $\mathcal{X}$ of random quantities is incoherent if there exist alternative numbers $\{Q(X) : X \in \mathcal{X}\}$, finitely many of the random quantities $X_1, \ldots, X_n$, positive numbers $a_1, \ldots, a_n$, and $\epsilon > 0$ such that

$$\sum_{i=1}^{n} a_i[X_i(t) - P(X_i)]^2 > \epsilon + \sum_{i=1}^{n} a_i[X_i(t) - Q(X_i)]^2, \text{ for all } t. \tag{7.1}$$

The previsions are coherent if they are not incoherent. The interpretation is that one will be subject to a loss equal to some multiple of the squared difference between one's prevision and the actual value of each random quantity. It would be incoherent to choose a set of previsions if there existed an alternative set (the $Q(X)$ numbers) which led to uniformly smaller loss. The quadratic loss function is often called *Brier score* in the elicitation literature.

We can extend this second definition to upper and lower previsions and then try to measure the extent of incoherence according to the second definition. We can call x_- a lower prevision for X if for all $a > 0$ and $y < x_-$, you prefer to lose $a(X-x_-)$ to losing $a(X-y)$. In this case, call all $y < x_-$ *too extreme for* X. Similarly, x_+ is an upper prevision for X if for all $a > 0$ and $y > x_+$, you prefer to lose $a(X-x_+)$ to losing $a(X-y)$. In this case all $y > x_+$ are *too extreme for* X. Now, a collection of upper and/or lower previsions $\{P(X) : X \in \mathcal{X}\}$ for a set $\mathcal{X}$ of random quantities is incoherent if there exist alternative numbers $\{Q(X) : X \in \mathcal{X}\}$, finitely many of the random quantities $X_1, \ldots, X_n$, positive numbers $a_1, \ldots, a_n$, and $\epsilon > 0$ such that for each X, $Q(X)$ is too extreme for X and (7.1) holds.

To see the connection between the two definitions of upper/lower prevision, we can assume that a bookie who prefers to lose $a(X-x)$ to losing $a(X-y)$ finds the "trade" $a(X-y) - a(X-x)$ favorable as a gain. This is in the same spirit as the calculation in Section 3. The net gain to the bookie is

$$a(X-y)^2 - a(X-x)^2 = 2a(x-y)\left(X - \frac{1}{2}[x+y]\right). \tag{7.2}$$

This is precisely in the form of a gamble that we have already considered. If x is an upper prevision in the sense of this section, then (7.2) holds for all $y > x$, and the coefficient on the right-hand side is negative. This makes $\alpha(X-z)$ acceptable for $\alpha < 0$ and $z > x$. Hence, x is an upper prevision in the sense of Section 2. If x is an upper prevision in the sense of Section 2, then the gamble on the right-hand side of (7.2) is favorable for all $y > x$ and x is an upper prevision in the sense of this section. The two definitions are equivalent. The same can be said of the definitions of lower prevision. The two definitions of incoherent are equivalent as well.

THEOREM 7. *A set of upper and/or lower previsions is incoherent by the definition of this section if and only if it is incoherent by the definition of Section 2.*

Now, try to measure the incoherence according to this new definition in a manner similar to what we have already done. If a bookie announces a lower prevision x_- for X, a gambler might offer the following trade in the same spirit as Section 3. The bookie pays the gambler $a(X - x_-)^2$ and the gambler pays the bookie $a(X - y)^2$, where $a > 0$ and $y < x_-$. Since the bookie prefers the first loss to the second, this should seem like a favorable trade to the bookie. The net gain to the bookie is $2a(x_- - y)(X - [x_- + y]/2)$. Since $y < x_-$, the coefficient is positive and the gamble corresponds to the fact that x_- is a lower prevision according to the definition in Section 2. We can define the escrow for a trade like the one in this paragraph to be the escrow corresponding to the gamble $2a(x_- - y)(X - [x_- + y]/2)$. Notice that $2a(x_- - y)$ is an arbitrary positive number and $[x_- + y]/2$ is an arbitrary number less than x_-. These two numbers can be chosen independently of each other by appropriate choice of y and a. So, the maximum guaranteed loss to the bookie or the minimum guaranteed profit to the gambler subject to the appropriate escrow being less than 1 will be the same as that derived in the other theorems in this paper.

8. Summary and some directions for additional work. In this article we introduce two indices of incoherence of previsions, based on the gambling framework of de Finetti (1974). When a bookie is incoherent, a gambler can choose a collection of gambles acceptable to the bookie that result in a sure loss to the bookie (and a sure gain to the gambler). That is, the gambler can make a *Dutch Book* against the bookie. As one index of incoherence in the bookie's previsions, we use the maximum guaranteed rate of loss to the bookie that the gambler creates through his/her choice of coefficients, relative to the bookie's escrow. For our second index, the bookie's incoherence is measured by the maximum guaranteed rate of profit the gambler creates relative to his/her own escrow. Throughout, we mean by "escrow" the sum over separate gambles of the maximum payment out that a given player faces in each gamble. In other words, "escrow" is the sum of what a player in this game must ante to cover each particular gamble, where the ante for each gamble is judged independently of all the other gambles.

In Section 2 we consider the first of these two indices applied to combinations of lower and upper previsions made by the bookie on finitely many random variables X_i, $i = 1, \ldots, n$. These one-sided offers from the bookie apply to restrict the gambler (respectively) to positive or negative coefficients. De Finetti's theory, then, is the special case where, for each X_i, the bookie has equal lower and upper previsions.

We focus attention on the case of events over a finite partition. That is, X_i is the indicator function for event A_i. Here, for example, if each pair of the gambler's lower and upper previsions (p_i and q_i, respectively) for A_i is coherent on its own, i.e., $0 \leq p_i \leq q_i \leq 1$, then incoherence arises because the sum s^- (s^+) of the lower (upper) previsions is greater (less) than 1. Theorem 2 reports the gambler's maximin strategy in this game, maximizing the guaranteed minimum rate of loss to the bookie in such a case. Simply, choose equal coefficients with the sign of the coefficient determined by which case of incoherence obtains. It is worth noting that, either as s^- increases above 1 or as s^+ decreases below 1, the maximum rate of loss grows. The more the sum of the bookie's lower or upper previsions depart from their coherent bound, 1, the greater is this index of incoherence.

In Section 3, we apply this idea to fix the degree of incoherence in a policy of hypothesis testing using a fixed alpha-level regardless of the sample size. We show that the index of incoherence taken with respect to the rate of loss of the bookie coincides with the extent to which the fixed alpha-level tests are dominated alternative tests (when these are suitably scaled).

In Section 4, we examine the second index of incoherence, maximizing the rate of profit, applied to the same gambling context as used in Section 2, betting on events in a (finite) partition. The gambler's maximin strategy is described in Theorem 4. When $s^+ < 1$ this strategy coincides with maximizing the first index too, but not so when $s^- > 1$. As pictured in Figure 1, the difference in the two indices leads to different strategies for the gambler precisely when $s^- > 1$ and the bookie's lower previsions lie outside a projection of the simplex that represents the coherent bound ($s^- = 1$) for lower previsions.

In Section 5 we extend our investigation to a context where a simple random variable X is defined on the same partition of events for which betting is already available. However, we limit our analysis to the de Finetti-styled case where all upper and lower previsions are equal, i.e. previsions exist. Theorem 5 reports the optimum choice of coefficients for maximizing the rate of loss. In particular, it shows when the gambler is advised to ignore the bookie's prevision p_X for X and, instead, to concentrate bets on the incoherence of the previsions p_i for the A_i . Not surprising, as $s = \sum_{i=1}^{n} p_i)$ departs more and more from the coherent value, $s = 1$, the gambler ignores p_X over ever wider intervals of possible values. Theorem 6 gives the maximum rate of profit, sure-gains relative to the gambler's escrow, that results from incoherence involving p_X for the special case where $s < 1$.

In Section 6, we apply the preceding results to the case of called-off bets. Let p_X be a prevision for the called-off bet on A_1 against A_2, given that $A_3 = (A_1 \cup A_2)^C$ fails. It is a familiar justification of Bayes' rule for conditional probability that when the agent's previsions are coherent, $p_X = p2/(p1 + p2)$. However, as Theorem 5 shows for $s \neq 1$ and as Theorem 6 shows when $s < 1$, for a fixed value of s the bookie minimizes both his/her rate of loss and rate of profit by choosing this "Bayes'-odds" value for p_X. In that case the gambler can do no better than to ignore p_X and apply the consequences of Theorems 2 and 4 to fix a maximin strategy.

Last, in Section 7 (Theorem 7) we relate the idea of incoherence of lower and upper previsions to de Finetti's alternative criterion of coherence, involving generalized Brier score. We explain how de Finetti's other criterion leads to the same two indices of incoherence as studied in this paper for incoherent previsions.

Where do these results take us? Lindley, Tversky and Brown (1979) wrote,

> The difficulties involved in applying standard measurement criteria of reliability and validity to the measurement of belief gives rise to the questions of how to evaluate and improve assessment of subjective probability.

Both questions are important, we agree.

The approach Lindley, Tversky and Brown (1979) took was to model a person's incoherent previsions as a less than perfect report of some ur-agent's coherent previsions. For them, an announced prevision is a transmitted message of a coherent prevision distorted by some variety of personal measurement error and bias. Rather than grounding our analysis on a statistical model of error in a "noisy" elicitation process, we propose instead assessing incoherent previsions directly through the same decision-theoretic considerations that are used to motivate coherence: Dutch Book. Hence, we introduce rates of loss and profit that result from incoherence in lower and upper previsions.

We go only a short way into the woods of incoherence with the analysis reported here. We give results of two assessments, using rates of loss and profit, for the limited context of bets on elements of a finite partition together with one prevision for a simple random variable defined on that same partition. Obviously, this is not a fully general assessment of incoherent previsions. Thus, one direction for research is to extend these findings by assessing these same two degrees of incoherence for increased numbers and varieties of previsions.

Such proposed generalization raises the question of how to compute the optimal Dutch-Bookmaking strategies. We note that, unfortunately, not always do we find closed form solutions. However, as shown in Theorem 1 for the bookie's escrow, the maximal $g(\alpha)$ is linear in α, while the constraint $h(\alpha)$ is piecewise linear in α. As in Nau (1992) - ours is the special case of his with equal, i.e., unrestricted so-called "confidence weights" - this results in a finite set of linear programming problems. Similarly, the gambler's perspective of Section 4 results in a (different) finite series of linear programming problems. How efficiently these problems can be solved remains an open theoretical question, we believe.

The second goal of Lindley, Tversky and Brown (1979) is to provide guidance for improving previsions - as distinct from the first goal of assessing them. The analysis in this paper does little more than hint at how a bookie with incoherent previsions might improve them. One idea stems from the following observation: when maximizing the bookie's rate of loss or when maximizing the gambler's rate of profit, sometimes the gambler localizes the bookie's incoherence. For instance, we found that when the bookie's called-off prevision for two events is close to the Bayes-like odds for those events, then the gambler's maximizing strategy ignores the called-off prevision entirely. In that sense, the incoherence is localized to the bookie's unconditional previsions. Can this phenomenon be used by the bookie to direct where to adjust the incoherence?

Relating to this, also,we note the work of Zynda (1996), who takes a qualitative approach to the problem. He focuses on the coherent sub-algebras of an incoherent assignment of probabilities, reflecting concerns about an agent's logical fallibility. That is, he locates those collections of probability assignments that constitute a coherent set. This, we think, is in the spirit of identifying maximally consistent sets of beliefs.

Last, as we illustrate with the case of hypothesis testing at a fixed alpha-level, we think that these results on degrees of incoherence offer yet another basis for a comparison of Bayesian with Classical statistical methods. Can we adapt our indices for rates of loss and profit to help gauge which classical but inadmissible procedures are nearly coherent, and which are far from coherent?

APPENDIX

A.1. Proof of Theorem 1. Define

$$\begin{aligned} g_1(\alpha_1,\ldots,\alpha_n,y_1,\ldots,y_n) &= \sup_{t\in T}\sum_{i=1}^{n}\alpha_i(X_i(t)-y_i), \\ h_1(\alpha_1,\ldots,\alpha_n,y_1,\ldots,y_n) &= -\sum_{i=1}^{n}\min\{0,\inf_{t\in T}\alpha_i(X_i(t)-y_i)\}. \end{aligned}$$

Then $g_1 = -G(\sum_{i=1}^{n}\alpha_i(X_i-y_i))$ and h_1 is the amount the bookie must escrow in order to gamble. Since the previsions are incoherent, there exists at least one combination Y

of gambles with $G(Y) > 0$ and strictly positive escrow, so we do not need to worry about the denominator of (2.2) being 0 in the vicinity of the supremum. Since

$$-g(\alpha_1,\ldots,\alpha_n) + g_1(\alpha_1,\ldots,\alpha_n,y_1,\ldots,y_n) \geq \\ h(\alpha_1,\ldots,\alpha_n) - h_1(\alpha_1,\ldots,\alpha_n,y_1,\ldots,y_n) > 0$$

whenever the y_i are on the proper sides of the x_i, and since $0 \leq -g_1/h_1 \leq 1$, it follows that the right-hand side of (2.2), call it ρ_0, is at least as large as ρ. Let $\epsilon > 0$ and let

$$\frac{-g(\alpha_{1,0},\ldots,\alpha_{n,0})}{h(\alpha_{1,0},\ldots,\alpha_{n,0})} > \rho_0 - \epsilon. \tag{A.1}$$

We will complete the proof by showing that there exist $y_1,\ldots,y_n$ all on the proper sides of the x_is such that

$$\frac{-g_1(\alpha_{1,0},\ldots,\alpha_{n,0},y_1,\ldots,y_n)}{h_1(\alpha_{1,0},\ldots,\alpha_{n,0},y_1,\ldots,y_n)} \geq \rho_0 - 2\epsilon. \tag{A.2}$$

Since g_1/h_1 is a continuous function of $(y_1,\ldots,y_n)$ in a neighborhood of $(x_1,\ldots,x_n)$ for fixed αs, there exist $y_1,\ldots,y_n$ such that each y_i is on the proper side of x_i and the ratio in (A.2) is within ϵ of the ratio in (A.1).

A.2. Proof of Theorem 2. Two lemmas are useful for the proof of Theorem 2 as well as later.

LEMMA 1. *Let $\{A_i\}_{i=1}^m$ be a collection of disjoint nonempty events with lower previsions $p_i \geq 0$ for $i = 1,\ldots,m$ such that $s^- = \sum_{i=1}^m p_i \geq 1$ and $m > 1$. Then $\rho = (s^- - 1)/s^-$ and $\alpha_j = 1/s^-$ for all j achieves the* inf *in (2.3). If $p_i > 0$ for all i and $s^- > 1$, then the $\alpha_1,\ldots,\alpha_m$ that achieve the* inf *in (2.3) are unique.*

PROOF. If $s^- = 1$, then the lower previsions are coherent and $\rho = 0$. In this case, all α_i equal to each other achieves this value. For the rest of the proof, assume that $s^- > 1$. In this case, we know that there exist $\alpha_1,\ldots,\alpha_m$ (in particular those given in the statement of the lemma) such that $g(\alpha_1,\ldots,\alpha_m) < 0$.

If $\{A_i\}_{i=1}^m$ do not form a partition, let $n = m+1$, $A_n = (\cup_{i=1}^m A_i)^C$, $\alpha_n = 0$ and $p_n = 0$. If $\{A_i\}_{i=1}^m$ do form a partition, then let $n = m$. In either case, the combination of gambles in this problem is

$$\sum_{i=1}^n \alpha_i(I_{A_i} - p_i) = \sum_{i=1}^n \alpha_i I_{A_i} - \sum_{i=1}^n \alpha_i p_i. \tag{A.3}$$

Let $c = \sum_{i=1}^n \alpha_i p_i$. Then $g(\alpha_1,\ldots,\alpha_n) = \max\{\alpha_1,\ldots,\alpha_n\} - c$. Clearly having all $\alpha_i = 0$ does not achieve the inf in (2.3), so assume that at least one $\alpha_i > 0$. In such cases, $h(\alpha_1,\ldots,\alpha_n) = \sum_{i=1}^m \alpha_i p_i = c > 0$. (Since $g < 0$, we cannot have $c = 0$.) Clearly, if $c < 1$, we can make g smaller (strictly smaller if all $p_i > 0$) by scaling up the α_i to make $c = 1$. So assume that $c = 1$. Now, $g = \max\{\alpha_1,\ldots,\alpha_m\} - 1$. We minimize g by making the largest α_i as small as possible subject to $\sum_{i=1}^m \alpha_i p_i = 1$. If the α_i are not all equal, then it is easy to see that we can lower the largest ones by raising the smallest ones while maintaining the constraint $\sum_{i=1}^m \alpha_i p_i = 1$. If $0 < p_i$ for all $i = 1,\ldots,m$, then this maneuver strictly lowers g. This implies that g is minimized by choosing all of the α_i equal to the same value, which must, by the constraint, be the value $1/\sum_{i=1}^m p_i$. Plugging this value for all

α_i into the formula for g yields the value of ρ stated in the lemma. Uniqueness in the case $0 < p_i$ for all $i = 1, \ldots, m$ follows from the series of strict decreases in g that occurred in the above discussion. □

The next lemma handles the case in which the sum of upper previsions is too small.

LEMMA 2. *Let $\{A_i\}_{i=1}^n$ be a collection of $n > 1$ disjoint nonempty events with upper previsions $0 \leq q_i$ and $s^+ = \sum_{i=1}^n q_i < 1$. Let*

$$g'(\alpha_1, \ldots, \alpha_n) = \sup_{t \in \cup_{i=1}^n A_i} \sum_{i=1}^n \alpha_i (I_{A_i}(t) - q_i).$$

Then

- *$\alpha_j = -1/(n - s^+)$ for all j will minimize g' subject to $h(\alpha_1, \ldots, \alpha_n) \leq 1$. Call the minimum value ρ'.*
- *If $\{A_i\}_{i=1}^n$ is a partition, then $\rho' = (1 - s^+)/(n - s^+)$ and $\alpha_j = -1/(n - s^+)$ for all j achieves the* inf *in (2.3).*

If all $q_i > 0$, the $\alpha_1, \ldots, \alpha_n$ that minimize g' are unique.

PROOF. We have $g'(\alpha_1, \ldots, \alpha_n) = \max\{\alpha_1, \ldots, \alpha_n\} - c$, where $c = \sum_{i=1}^n \alpha_i q_i$, and $g' = g$ in the partition case. Let $\alpha_j = \max\{\alpha_1, \ldots, \alpha_n\}$, so that $g'(\alpha_1, \ldots, \alpha_n) = \alpha_j - c$. Then

$$g'(\alpha_j, \ldots, \alpha_j) = \alpha_j - c + \sum_{i \neq j} (\alpha_i - \alpha_j) q_i \leq g'(\alpha_1, \ldots, \alpha_n), \tag{A.4}$$

since $\alpha_i \leq \alpha_j$ for all $i \neq j$ and $q_i \geq 0$. Also,

$$h(\alpha_j, \ldots, \alpha_j) = h(\alpha_1, \ldots, \alpha_n) + \sum_{i \neq j} (\alpha_i - \alpha_j)(1 - q_i) \leq h(\alpha_1, \ldots, \alpha_n),$$

since $\alpha_i \leq \alpha_j$ for all $i \neq j$ and $1 - q_i \geq 0$. It follows that g' is minimized (subject to $h \leq 1$) by setting all α_i equal to each other and rescaling them to make $h = 1$. Uniqueness in the case $0 < q_i$ for all $i = 1, \ldots, m$ follows from the fact that the inequality in (A.4) is strict if all $q_i > 0$ and the α_i are not all equal. □

PROOF OF THEOREM 2. For part 1, suppose that we contemplate a combination of gambles

$$\sum_{i=1}^n \alpha_i (I_{A_i} - p_i) + \sum_{i=1}^n \beta_i (I_{A_i} - q_i), \tag{A.5}$$

with all $\alpha_i \geq 0$ and all $\beta_i \leq 0$. We will show that, in order to maximize ρ, we must have all $\beta_i = 0$. Together with Lemma 1, this will imply that the conclusion from part 1 holds. Although not acceptable, consider the following alternate combination of gambles

$$\sum_{i=1}^n \alpha_i (I_{A_i} - p_i) + \sum_{i=1}^n \beta_i (I_{A_i} - p_i). \tag{A.6}$$

The gamble (A.6) equals (A.5) minus $\sum_{i=1}^n \beta_i (p_i - q_i)$ in all circumstance. This value is nonnegative, and it also equals the additional escrow required for (A.6) over that required for (A.5). It follows that the maximum value for ρ based on gambles of the form (A.6) will be at least as close to 1 as (hence at least as large as) the value based on (A.5).

We will prove that all $\beta_i = 0$ is necessary to maximize ρ based on (A.6), hence it is also necessary to maximize ρ based on (A.5) because the two gambles are the same when all $\beta_i = 0$. Rewrite (A.6) as

$$\sum_{i=1}(\alpha_i + \beta_i)(I_{A_i} - p_i). \tag{A.7}$$

This gamble has the same payoff as (A.6) but its escrow is no larger because $(\alpha_i + \beta_i)p_i \leq \alpha_i p_i$ whenever $\alpha_i + \beta_i > 0$ and $-(\alpha_i + \beta_i)(1 - p_i) \leq -\beta_i(1 - p_i)$ whenever $\alpha_i + \beta_i < 0$. It follows that the maximum value for ρ based on gambles of the form (A.7) will be at least as large as the value based on (A.6). We prove below that all $\alpha_i + \beta_i \geq 0$ is necessary to maximize ρ based on (A.7), hence all $\beta_i = 0$ is also necessary to maximize ρ based on (A.6) because the two gambles are the same when all $\beta_i = 0$.

For convenience, define $\gamma_i = \alpha_i + \beta_i$. Suppose first that all $\gamma_i < 0$. Let $c = \sum_{i=1}^n \gamma_i p_i$. Then $g(\gamma_1, \ldots, \gamma_n) = \max\{\gamma_1, \ldots, \gamma_n\} - c$. Since $\sum_{i=1}^n p_i > 1$, $c < \max\{\gamma_1, \ldots, \gamma_n\}$. It follows that $g(\gamma_1, \ldots, \gamma_n) > 0$ which cannot provide a maximum for ρ. Hence at least one $\gamma_i \geq 0$. Next, suppose that for some j $\gamma_j < 0$, $g(\gamma_1, \ldots, \gamma_n) < 0$ and $h(\gamma_1, \ldots, \gamma_n) \leq 1$. Since at least one $\gamma_i \geq 0$, we know that $\gamma_j < \max\{\gamma_1, \ldots, \gamma_n\}$. We show next that replacing γ_j by 0 will not raise the value of g and it will not violate the constraint $h \leq 1$. Without loss of generality, let $\gamma_1 < 0$. Then $c' = \sum_{i=2}^n \gamma_i p_i \geq c$ and

$$g(0, \gamma_2, \ldots, \gamma_n) = \max\{0, \gamma_2, \ldots, \gamma_n\} - c' = g(\gamma_1, \ldots, \gamma_n) - c' + c.$$

Since $c' \geq c$, we have not raised g by replacing γ_1 by 0. Also, if $p_1 > 0$, g strictly decreases. Clearly $h \leq 1$ still holds because setting $\gamma_1 = 0$ is equivalent to deleting the first gamble. It follows that we cannot have a single $\gamma_i < 0$ if we wish to maximize ρ.

For part 2, we can argue as above that the maximum ρ achievable by a gamble of the form (A.5) is no larger than that available by a gamble of the form

$$\sum_{i=1}(\alpha_i + \beta_i)(I_{A_i} - q_i), \tag{A.8}$$

and that the two maxima are the same if maximizing (A.8) requires all $\alpha_i + \beta_i \leq 0$. For convenience, define $\gamma_i = \alpha_i + \beta_i$. Let $c = \sum_{i=1}^n \gamma_i q_i$. Then $g(\gamma_1, \ldots, \gamma_n) = \max\{\gamma_1, \ldots, \gamma_n\} - c$. Suppose first that all $\gamma_j > 0$. Since $\sum_{i=1}^n q_i < 1$, we have $c < \max\{\gamma_1, \ldots, \gamma_n\}$ and $g > 0$, which cannot provide a maximum for ρ. Hence at least one $\gamma_i \leq 0$. Next, suppose that some $\gamma_j > 0$, $g'(\gamma_1, \ldots, \gamma_n) < 0$ and $h(\gamma_1, \ldots, \gamma_n) \leq 1$. We show now that replacing all of the positive γ_j by 0 will not raise the value of g and it will not violate the constraint $h \leq 1$. Without loss of generality, let $\gamma_{k+1}, \ldots, \gamma_n > 0$ and $\gamma_1, \ldots, \gamma_k \leq 0$. Let $c' = \sum_{i=1}^k \gamma_i q_i = c - \sum_{i=k+1}^n \gamma_i q_i$ so that

$$g(\gamma_1, \ldots, \gamma_k, 0, \ldots, 0) \leq -c' = \sum_{i=k+1}^{n} \gamma_i q_i - c < g(\gamma_1, \ldots, \gamma_n),$$

because $\sum_{i=k+1}^n \gamma_i q_i < \max\{\gamma_{k+1}, \ldots, \gamma_n\}$. So, we have lowered g by replacing the positive γ_js by 0. Since replacing $\gamma_{k+1}, \ldots, \gamma_n$ by 0 is equivalent to deleting a few gambles from the collection, $h \leq 1$ still holds. It follows that we cannot have a single $\gamma_i > 0$ if we wish to maximize ρ. Lemma 2 completes the proof of part 2. □

A.3. Proof of Theorem 3. A lemma is useful before giving the proof.

Lemma 3. *Let $\{A_i\}_{i=1}^n$ be a collection of events with upper previsions $q_i \le 0$ with at least one $q_i < 0$. Suppose that $\cap_{i=1}^n A_i^C = B \ne \emptyset$ and no $A_i = \emptyset$. Let $q_n = \min\{q_1, \ldots, q_n\}$. Then $\rho = -q_n/(1-q_n)$ and a gamble that achieves this value has all $\alpha_i = 0$ except for $\alpha_n = 1/(1-q_n)$.*

Proof. Consider an acceptable gamble $\sum_{i=1}^n \alpha_i(I_{A_i} - q_i)$ with all $\alpha_i \le 0$. In this case, $h(\alpha_1, \ldots, \alpha_n) = -\sum_{i=1}^n \alpha_i(1-q_i) = c$. If $c < 1$ and $g(\alpha_1, \ldots, \alpha_n) < 0$, then we can make g even smaller by scaling up the α_i to make $c = 1$. So, we will assume $c = 1$. Since all $\alpha_i \le 0$, the $\sup_{t \in T}$ occurs for $t \in B$ and $g(\alpha_1, \ldots, \alpha_n) = -\sum_{i=1}^n \alpha_i - 1$. Making this as small as possible, subject to $h = 1$, is effected by setting all $\alpha_i = 0$ except the one corresponding to the largest $1 - q_i$, namely q_n. Set $\alpha_n = -1/(1-q_n)$ to make $h = 1$, and the value of g becomes $q_n/(1-q_n) = -\rho$. $\square$

Proof of Theorem 3. We can express g as

$$g(\alpha_1, \ldots, \alpha_n) = \max\{\alpha_1, \ldots, \alpha_n\} - c,$$

where $c = \sum_{i=1}^n \alpha_i q_i$. For the first part, it is easy to see that $\sum_{i=k+1}^n \alpha_i(I_{A_i}(t) - q_i) < 0$ for all t and it is largest for $t \in \cup_{i=1}^k A_i = B$. In order for the addition of $\sum_{i=1}^k \alpha_i(I_{A_i} - p_i)$ to make the maximum smaller, it is necessary that $\sum_{i=1}^k \alpha_i(I_{A_i}(t) - p_i) < 0$ for all $t \in B$. That is, let

$$g'(\alpha_1, \ldots, \alpha_k) = \max\{\alpha_1, \ldots, \alpha_k\} - c, \tag{A.9}$$

where $c = \sum_{i=1}^k \alpha_i p_i$. Then $g' < 0$ is necessary in order for the additional gambles to lower g. It is also necessary that $\max\{\alpha_{k+1}, \ldots, \alpha_n\} \le c$ so that the $\sup_{t \notin B}$ not increase too much. These two conditions are, in fact, necessary and sufficient, and they can be combined into

$$\max\{\alpha_1, \ldots, \alpha_n\} \le \sum_{i=1}^k \alpha_i p_i = c. \tag{A.10}$$

First, suppose that $p < 1$. Suppose that some $\alpha_i > 0$ for $i \le k$. Since $q < 1$, we have $c < \max\{\alpha_1, \ldots, \alpha_k\}$ which violates (A.10), so we must have $\alpha_i \le 0$ for all i. Since the p_i are lower previsions, we must have all $\alpha_i = 0$. This makes the combination of gambles

$$\sum_{i=k+1}^n \alpha_i(I_{A_i} - q_i).$$

Lemma 3 applies to such a combination because $q_i < 0$ for all $i > k$ and $\cap_{i=k+1}^n A_i^C = B \ne \emptyset$. The result follows since $p - 1 < 0$.

Next, suppose that $p \ge 1$. Since the p_i are lower previsions we need $\alpha_i \ge 0$ for $i \le k$. The escrow is then $h(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^k \alpha_i p_i - \sum_{i=k+1}^n \alpha_i(1-p_i)$. First fix $\alpha_{k+1}, \ldots, \alpha_n$. Since $\alpha_i \ge 0$ for $i \le k$, the highest value of the combination of gambles occurs on some A_i for $i \le k$. For this reason, we can minimize g by minimizing g' from (A.9) subject to

$$\sum_{i=1}^k \alpha_i p_i \le 1 + \sum_{i=k+1}^n \alpha_i(1-q_i). \tag{A.11}$$

Call the number on the right-hand side of (A.11) h'. Lemma 1 says that minimizing g' subject to $\sum_{i=1}^k \alpha_i p_i \le h'$ is effected by letting $\alpha_j = h'/\sum_{i=1}^k p_i$ for $j = 1, \ldots, k$. This means that, all α_i for $i \le k$ must be the same number $\alpha \ge 0$ in order to minimize g. This makes the portion of the escrow for the first k bets equal to $\sum_{i=1} \alpha p_i = \alpha p$, which is what the escrow would be if the gamble had been $\alpha(I_B - p)$. Let $q = 1 - p$ so that $\alpha(I_B - p) = -\alpha(I_{B^C} - q)$ and $q \leq 0$. This implies that we can act as if B^C had been assigned an upper prevision of q. So, our combination of bets is $-\alpha(I_{B^C} - q) + \sum_{i=k+1}^{n} \alpha_i(I_{A_i} - q_i)$. Now, apply Lemma 3 to the collection of events $B^C, A_{k+1}, \ldots, A_n$ each of which has nonpositive upper prevision.

For the second part (that is, $k = 0$), assume without loss of generality that $\sum_{i=1}^{n} \alpha_i(1 - q_i) = -1$. Then $c = 1 + \sum_{i=1}^{n} \alpha_i$ and $g(\alpha_1, \ldots, \alpha_n)$ equals minus the sum of all the α_i except the largest one then minus 1. We can make the sum of all but the largest α_i larger (hence make g smaller) by raising all α_i that are strictly less than the largest one up to the value of the largest one. The effect of this on the escrow is to lower the escrow. When we rescale the α_i to make the escrow equal to 1, g goes down again. Hence the minimum of g occurs with all α_i equal to the same negative value, which must be the value given in the statement. □

A.4. Proof of Theorem 4. We will prove part 1 under the assumption that only upper previsions are announced. As in the proof of Theorem 2, it is easy to argue that it is impossible to improve ψ by including gambles based on the lower previsions. We have

$$\begin{aligned} g(\alpha_1, \ldots, \alpha_n) &= \max\{\alpha_1, \ldots, \alpha_n\} - c, \\ h'(\alpha_1, \ldots, \alpha_n) &= -c, \end{aligned}$$

where $c = \sum_{i=1}^{n} \alpha_i q_i$. It is clear that $h' = 0$ implies $g = 0$, which is not the smallest possible value. If $0 < h' < 1$ we can scale up h' and make g smaller, so assume that $c = -1$. This makes g equal to the largest α_i plus 1. Set $\beta_i = -\alpha_i$ for each i and maximize $\min\{\beta_1, \ldots, \beta_n\}$ subject to $\sum_{i=1}^{n} \beta_i p_i = 1$. If the β_i are not all equal, then we can raise the lowest ones and lower the highest ones while maintaining the constraint. It follows that g is minimized by setting all α_j equal to $-1/s^+$.

We will prove part 2 for the case in which only lower previsions are announced. As in the proof of Theorem 2, it can be shown that no improvement in ψ is possible by basing gambles on the upper previsions. We have

$$\begin{aligned} g(\alpha_1, \ldots, \alpha_n) &= \max\{\alpha_1, \ldots, \alpha_n\} - c, \\ h'(\alpha_1, \ldots, \alpha_n) &= \sum_{i=1}^{n} \alpha_i - c, \end{aligned}$$

where $c = \sum_{i=1}^{n} \alpha_i p_i$. It is clear that $h' = 0$ implies $g = 0$, which is not the smallest possible value. If $0 < h' < 1$ we can scale up h' and make g smaller, so assume that $c = \sum_{i=1}^{n} \alpha_i - 1$. This makes g equal to 1 minus the sum of all α_i except the largest one. Minimizing g is equivalent to maximizing the sum of all but the largest α_i. First, we show that if the α_i are not ordered from smallest to largest, then we make the sum larger by rearranging the α_i from smallest to largest. For example, if $i < j$ but $\alpha_i > \alpha_j$, then $(1 - p_i)\alpha_i + (1 - p_j)\alpha_j \geq (1 - p_i)\alpha_j + (1 - p_j)\alpha_i$. Switching α_i and α_j makes h' no larger without affecting g, so we can scale up all the α_i to make $h' = 1$ and lower g. So $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ provides a value of g at least as small as what we started with.

Clearly, we can lower α_n and raise α_{n-1} until they are equal while maintaining the constraint. This will only lower g (unless $\alpha_n = \alpha_{n-1}$ already). Suppose that $\alpha_n = \alpha_{n-1} = \cdots = \alpha_{n-k+1} > \alpha_{n-k}$ for some $k > 1$. Consider lowering all of these by ϵ while raising α_{n-k} by δ and maintaining the constraint. This makes $\delta = \epsilon(k - p_n - \cdots - p_{n-k+1})/(1 - p_{n-k})$.

The change in g is $(k-1)\epsilon - \delta$. This is positive if and only if $(k-1)(1-p_{n+k}) < k - p_n - \cdots - p_{n-k+1}$, which is equivalent to $\sum_{i=n-k+1}^{n} p_i > 1 + (k-1)p_{n-k}$ or $k \geq k^*$. So this process will not raise g unless it continues through $k = k^*$ or beyond. Once $k \geq k^*$ the same reasoning shows that raising $\alpha_n, \ldots, \alpha_{n-k^*+1}$ and lowering α_{n-k} to 0 while maintaining the constraint will lower (or at least not raise) g. This process will lead to the same result no matter what is the initial configuration of $\alpha_1, \ldots, \alpha_n$. Hence, g is minimized by setting $\alpha_{n-k^*+1}, \ldots, \alpha_n = 1/\sum_{i=n-k^*+1}^{n}(1-p_i)$ and $\alpha_1, \ldots, \alpha_{n-k^*} = 0$. Also, we get $\psi = [\sum_{i=n-k^*+1}^{n} p_i - 1]/\sum_{i=n-k^*+1}^{n}(1-p_i)$. previsions.

A.5. Proof of Theorem 5. If $\alpha_X = 0$, then Theorem 2 will apply, so begin by assuming that $|\alpha_X| = 1$. If it turns out that α_X should be 0, we will discover this by noticing that the maximum rate of guaranteed loss occurs for infinite values of the α_i.

If $\alpha_X = 1$, the escrow for (5.2) will be

$$-x_{(1)} + p_X + \eta - \lambda, \tag{A.12}$$

where $\eta = \sum_{i=1}^{n} \alpha_i p_i$ and $\lambda = \sum_{i=1}^{n} \min\{\alpha_i, 0\}$. If $\alpha_X = -1$, the escrow will be

$$x_{(n)} - p_X + \eta - \lambda. \tag{A.13}$$

The rate of loss for the gamble (5.2) is the smallest value of minus (5.2) divided by (A.12) or (A.13) depending on the sign of α_X. The values of minus (5.2) when A_i occurs divided by (A.12) and (A.13) equal

$$\frac{x_i - \alpha_i + \eta - p_X}{x_{(n)} - p_X + \eta - \lambda}, \qquad \text{if } \alpha_X = -1, \tag{A.14}$$

$$\frac{-x_i - \alpha_i + \eta + p_X}{-x_{(1)} + p_X + \eta - \lambda}, \qquad \text{if } \alpha_X = 1. \tag{A.15}$$

The denominators in both (A.14) and (A.15) are positive because they are escrows.

First, consider (A.14) with $\alpha_X = -1$. In this case, the rate of loss for (5.2) will be the value of (A.14) for that i such that the numerator is smallest. That is, let j be such that $x_j - \alpha_j = \min\{x_1 - \alpha_1, \ldots, x_n - \alpha_n\}$, and assume that the numerator of (A.14) is positive for all i. Then (A.14) with $i = j$ is the rate of loss. For each $i \neq j$, we can raise α_i to α_i' so that $x_i - \alpha_i' = x_j - \alpha_j$. The effect of this is to raise η by $p_i(\alpha_i' - \alpha_i)$ and to raise λ by $\min\{0, \alpha_i'\} - \min\{0, \alpha_i\}$. The net effect on (A.14) (with $i = j$) is to increase the numerator more than the denominator. Since the ratio is at most 1, the ratio increases. Hence the gamble with highest rate of loss will satisfy $x_i - \alpha_i = c$ for all i and some c. This makes $\eta = \mu - cs$ and (A.14) becomes

$$r_-(c) = \frac{-\delta + c(1-s)}{-\delta + x_{(n)} - cs - \lambda}. \tag{A.16}$$

The condition that the numerators all be positive becomes $c(1-s) > \delta$.

Next, consider (A.15) with $\alpha_X = 1$. In this case, the rate of loss for (5.2) will be the value of (A.15) for that i such that the numerator is smallest. That is, let j be such that $-x_j - \alpha_j = \min\{-x_1 - \alpha_1, \ldots, -x_n - \alpha_n\}$, and assume that the numerator of (A.14) is positive for all i. Then (A.15) with $i = j$ is the rate of loss. For each $i \neq j$, we can raise α_i to α_i' so that $-x_i - \alpha_i' = -x_j - \alpha_j$. The effect of this is to raise η by $p_i(\alpha_i' - \alpha_i)$ and to raise λ by $\min\{0, \alpha_i'\} - \min\{0, \alpha_i\}$. The net effect on (A.15) (with $i = j$) is to increase the numerator more than the denominator. Since the ratio is at most 1, the ratio increases. Hence the gamble with highest rate of loss will satisfy $-x_i - \alpha_i = c$ for all i and some c. This makes $\eta = -\mu - cs$ and (A.15) becomes

$$r_+(c) = \frac{\delta + c(1-s)}{\delta - x_{(1)} - cs - \lambda}. \tag{A.17}$$

The condition that the numerators all be positive becomes $c(1-s) > -\delta$.

We need to find the value of c that makes $\max\{r_-(c), r_+(c)\}$ as large as possible. Each of the functions r_- and r_+ is the ratio of a linear function to a strictly positive continuous piecewise linear function. This makes the functions continuous and piecewise differentiable. Because the derivative of the ratio of a linear function to a positive linear function never changes sign, we know that the derivatives of r_- and r_+ can only change sign at the places where the denominator changes from one linear function to another one. The derivative of $(ax+b)/(dx+f)$ is $(af - bd)/(dx+f)^2$. Using this formula, we can write the derivatives for those c such that λ is the sum of k terms for $k = 0, \ldots, n$ as follows:

$$\begin{aligned} r_-'(c) &= \frac{\delta(k-1) + (s-1)[-x_{(n)} + \sum_{i=1}^{k} x_{(i)}]}{D_-^2}, \\ r_+'(c) &= \frac{-\delta(k-1) + (1-s)[-x_{(1)} + \sum_{i=n-k+1}^{n} x_{(i)}]}{D_+^2}, \end{aligned} \tag{A.18}$$

where D_- and D_+ are the denominators of r_- and r_+ respectively. These denominators do not affect the signs of the derivatives, so they will not be of interest to us here. How these derivatives change sign depends on s, so the remainder of the argument will be divided according to the value of s.

For $s < 1$, we must have $c > \delta/(1-s)$ in order to be able to use $\alpha_X = -1$ and $c > -\delta/(1-s)$ in order to use $\alpha_X = 1$. The derivatives from the right and from the left of r_- and r_+ must be positive at these points respectively because the functions are continuous and they are crossing from strictly below 0 to strictly above 0. When c crosses over $x_{(i)}$, the numerator of the derivative of r_-' changes by $\delta + (s-1)x_{(i)}$. These changes get smaller as i goes up (i.e. as c goes up). Hence r_-' can change sign from positive to negative at most once. Similarly, the change in r_+ when c crosses $-x_{(n-i+1)}$ is $(1-s)x_{(n-i+1)} - \delta$. These changes get smaller as i goes up (i.e. as c goes up). Hence r_+' can change sign from positive to negative at most once as well. Hence, the key to locating the maximum of each of these functions is the sign of the derivative at $c = \infty$, that is when $k = n$ in (A.18). These will be the signs of $(s-1)\sum_{i=1}^{n-1} x_{(i)} + \delta(n-1)$ and $(1-s)\sum_{i=2}^{n} x_{(i)} - \delta(n-1)$ respectively. At most one of these can be negative, so at most one of r_+ and r_- can have its maximum at a finite value of c. Since they both have the same limit as $c \to \infty$, it follows that the larger of the two maxima will be the one that occurs at a finite value, if either does. Otherwise, the maximum occurs at $c = \infty$, which corresponds to the result of Theorem 2.

For $s > 1$, we must have $c < -\delta/(s-1)$ in order to be able to use $\alpha_X = -1$ and $c < \delta/(s-1)$ in order to use $\alpha_X = 1$. The derivatives from the right and from the left of r_- and r_+ must be negative at these points respectively because the functions are continuous and they are crossing from strictly above 0 to strictly below 0. When c crosses over $x_{(i)}$, the numerator of the derivative of r'_- changes by $\delta + (s-1)x_{(i)}$. These changes get larger as i goes up (i.e. as c goes up). Hence r'_- can change sign from positive to negative at most once. Similarly, the change in r_+ when c crosses $-x_{(n-i+1)}$ is $(1-s)x_{(n-i+1)} - \delta$. These changes get larger as i goes up (i.e. as c goes up). Hence r'_+ can change sign from positive to negative at most once as well. Hence, the key to locating the maximum of each of these functions is the sign of the derivative at $c = -\infty$, that is when $k = 0$ in (A.18). These will be the signs of $-(s-1)x_{(n)} - \delta$ and $(s-1)x_{(1)} + \delta$ respectively. At most one of these can be positive, so at most one of r_+ and r_- can have its maximum at a finite value of c. Since they both have the same limit as $c \to -\infty$, it follows that the larger of the two maxima will be the one that occurs at a finite value, if either does. Otherwise, the maximum occurs at $c = -\infty$, which corresponds to the result of Theorem 2. Also, notice that the numerators in (A.18) when $k = 1$ are both equal to $(1-s)(x_{(n)} - x_{(1)}) < 0$. This means that, if r_- or r_+ has a maximum at a finite value, that finite value must be the lowest possible value, namely $c = x_{(1)}$ if $\alpha_X = -1$ and $c = -x_{(n)}$ if $\alpha_X = 1$.

If $s = 1$, then (A.16) becomes $-\delta/(-\delta + x_{(n)} - c - \lambda)$ and (A.17) becomes $\delta/(\delta - x_{(1)} - c - \lambda)$. Since the denominators are positive, we see that we must have $\alpha_X = -1$ if $\delta < 0$ and $\alpha_X = 1$ if $\delta > 0$ in order for the numerators to be positive. Since the numerator and denominator are both positive in both ratios, the appropriate ratio will be largest when $c + \lambda$ is largest. First, consider the $\alpha_X = -1$ case. In this case, $\lambda = 0$ if $c < x_{(1)}$ since then all $\alpha_i \geq 0$, and then $c + \lambda = c < x_{(1)}$. If $x_{(1)} \leq c \leq x_{(2)}$, then $\lambda = x_{(1)} - c$ and $c + \lambda = x_{(1)}$. If $c > x_{(2)}$, then $\lambda \leq x_{(1)} + x_{(2)} - 2c$, and $\lambda + c \leq x_{(1)} + x_{(2)} - c < x_{(1)}$. It follows that, (A.16) is largest for $c \in [x_{(1)}, x_{(2)}]$, and (A.16) then equals $-\delta/(-\delta + x_{(n)} - x_{(1)})$. Next, consider the $\alpha_X = 1$ case. Clearly, $\lambda = 0$ if $c < -x_{(n)}$ since then all $\alpha_i \geq 0$, and then $c + \lambda = c < -x_{(n)}$. If $-x_{(n)} \leq c \leq -x_{(n-1)}$, then $\lambda = -x_{(n)} - c$ and $c + \lambda = -x_{(n)}$. If $c > -x_{(n-1)}$, then $\lambda \leq -x_{(n)} - x_{(n-1)} - 2c$, and $\lambda + c \leq -x_{(n)} - x_{(n-1)} - c < -x_{(n)}$. It follows that (A.17) is largest for $c \in [-x_{(n)}, -x_{(n-1)}]$, and (A.17) then equals $\delta/(\delta + x_{(n)} - x_{(1)})$.

A.6. Proof of Theorem 6. The proof will require a few lemmas. We will use the following facts in all of the lemmas as well as in the main proof. If $\alpha_X = 1$, the escrow for (5.4) will be

$$x_n - p_X - \eta + \lambda, \tag{A.19}$$

where $\eta = \sum_{i=1}^n \alpha_i p_i$ and $\lambda = \sum_{i=1}^n \max\{\alpha_i, 0\}$. If $\alpha_X = -1$, the escrow will be

$$-x_1 + p_X - \eta + \lambda, \tag{A.20}$$

The rate of profit for the gamble (5.4) is the smallest value of minus (5.4) divided by (A.19) or (A.20) depending on the sign of α_X. The values of minus (5.4) when A_i occurs divided by (A.19) and (A.20) equal

$$\frac{x_i - \alpha_i + \eta - p_X}{-x_1 + p_X - \eta + \lambda}, \qquad \text{if } \alpha_X = -1, \tag{A.21}$$

$$\frac{-x_i - \alpha_i + \eta + p_X}{x_n - p_X - \eta + \lambda}, \qquad \text{if } \alpha_X = 1. \tag{A.22}$$

The denominators in (A.21) and (A.22) are positive because they are escrows.

Lemma 4. *Let $\alpha_1, \ldots, \alpha_n$ be given. Suppose that $\alpha_j < 0$ and the minimum value of the ratio, either (A.21) or (A.22), occurs at $i \neq j$. Then replacing α_j by* $\min\{0, \alpha_X(x_i - x_j) + \alpha_i\}$ *makes the minimum ratio no smaller.*

PROOF. Note that raising a single $\alpha_j < 0$ lowers the denominator and raises the numerator of either ratio (A.21) or (A.22) for $i \neq j$. The value $\min\{0, \alpha_X(x_i - x_j) + \alpha_i\}$ is the largest to which α_j can be raised without either changing the location of the minimum ratio or violating $\alpha_j < 0$. □

COROLLARY 3. *Given $\alpha_1, \ldots, \alpha_n$ with at least one $\alpha_i < 0$, there exists c such that*

- *the minimum ratio is made no smaller by moving each $\alpha_i < 0$ to $-\alpha_X x_i - c$, and*
- *the minimum ratio occurs for all i such that $\alpha_i < 0$.*

LEMMA 5. *Let $\alpha_1 \ldots, \alpha_n$ be given. Suppose that all negative α_i are configured as in Corollary 3. Then $\alpha_j \geq 0$ implies*

$$\alpha_{j'} \geq 0 \begin{cases} \textit{for all } j' > j & \textit{if } \alpha = -1, \\ \textit{for all } j' < j & \textit{if } \alpha_X = 1. \end{cases}$$

PROOF. Suppose that $\alpha_X = -1$ and $j' > j$ but $\alpha_{j'} < 0$. Then, the numerator of (A.21) for $i = j'$ is strictly greater than the numerator of (A.21) for $i = j$. Similarly if $\alpha_X = 1$ and $j' < j$ but $\alpha_{j'} < 0$, then numerator of (A.22) for $i = j'$ is strictly greater than the numerator of (A.22) for $i = j$. These violate the condition that the minimum ratio occurs whenever $\alpha_i < 0$. □

PROOF OF THEOREM 6. First, note that $v_-(n) = \delta - (1-s)x_1$ and $v_+(n) = (1-s)x_n - \delta$, so $k^* \leq n$ in both cases 1 and 2 in the theorem. Also, it is impossible that both $v_-(n) < 0$ and $v_+(n) < 0$ since this would imply $x_n < \delta/(1-s) < x_1$.

Suppose that at least one $\alpha_i > 0$. Consider what happens if we change all α_i to $\alpha_i + c$. The numerators of (A.21) and (A.22) change by $c(s-1)$ and the denominators change by $c(k_+ - s)$, where k_+ is the number of $\alpha_i > 0$. Since $s \leq 1$, $c < 0$ will make the numerators go up (or stay the same) while the denominators go down or stay the same so long as $k_+ \geq s$. It follows that shifting all α_i to $\alpha_i - \max_i \alpha_i$ makes the rate of guaranteed profit no smaller. If $s = 1$ the numerators never change with c and the denominator also stays fixed so long as there is exactly one positive α_i and exactly one positive $\alpha_i + c$. This will establish the additional claim about $s = 1$ once the remainder of the proof is completed. We will assume that $\alpha_i \leq 0$ for all i for the rest of the proof. They cannot all be 0 since p_X is itself coherent, so there exists at least one $\alpha_i < 0$. Let $\alpha_X \in \{1, -1\}$.

According to Corollary 3, there exists c such that for each i with $\alpha_i < 0$, $\alpha_i = -\alpha_X x_i - c$. Let k_- be the number of strictly negative α_i. We can rewrite the minimum ratios as

$$\frac{c(1 - s_{-,k_-}) + \sum_{i=1}^{k_-} x_i p_i - p_X}{-x_1 + p_X + c s_{-,k_-} - \sum_{i=1}^{k_-} x_i p_i}, \qquad \text{if } \alpha_X = -1,$$
$$\frac{c(1 - s_{+,k_-}) - \sum_{i=n-k_-+1}^{n} x_i p_i + p_X}{x_n - p_X + c s_{+,k_-} + \sum_{i=n-k_-+1}^{n} x_i p_i}, \qquad \text{if } \alpha_X = 1.$$

Ignoring strictly positive denominators, the derivatives of these minimum ratios with respect to c over those intervals such that the negative α_i stay negative and the minimum ratios stay in the same places can be written as $v_-(k_-)$ and $v_+(k_-)$ respectively. Note that $v_-(k)$ and $v_+(k)$ decrease as k increases, and they are nonnegative when $k = 1$. Since it is impossible that both $v_-(n) < 0$ and $v_+(n) < 0$, we must have $\alpha_X = -1$ if $v_-(n) < 0$

and $\alpha_X = 1$ if $v_+(n) < 0$. Suppose that $v_-(n) < 0$ (that is $\delta < (1-s)x_1$). For each $k_- \geq k^*$, we raise the minimum ratio by choosing c equal to the smallest possible value, thereby shifting all negative α_i so that the largest one is 0 and we make k_- decrease by 1. For each $k_- < k^* - 1$, we can increase the minimum ratio by shifting all negative α_i down until the minimum ratio also occurs at the i corresponding to $\alpha_{k_-+1} = 0$. We can consider this configuration as an extreme version of the case with k_- one larger. Hence, we are driven to the configuration with exactly $k^* - 1$ strictly negative α_i. That the α_i must be configured as stated follows from Corollary 3 and Lemma 5. A similar argument applies if $\delta > (1-s)x_n$. If $(1-s)x_1 \leq \delta \leq (1-s)x_n$, then every configuration with all $\alpha_i \leq 0$ is improved by making all $\alpha_i < 0$ and then letting c be as large as possible. Since $c \to \infty$ is allowed, this means that the $\alpha_i \to -\infty$ and this is equivalent to $\alpha_X = 0$, so Theorem 4 gives the maximum rate of profit. □

A.7. Proof of Theorem 7. Suppose that a set $\{P(X) : X \in \mathcal{X}\}$ of upper and/or lower previsions is incoherent by the definition of this section. Let $\{Q(X) : X \in \mathcal{X}\}$, $X_1, \ldots, X_n$, $a_1, \ldots, a_n$ and $\epsilon > 0$ be such that

$$\sum_{i=1}^{n} a_i[X_i(t) - P(X_i)]^2 > \epsilon + \sum_{i=1}^{n} a_i[X_i(t) - Q(X_i)]^2, \text{ for all } t.$$

Let $\alpha_i = 2a_i[P(X_i) - Q(X_i)]$ and $y_i = [P(X_i) + Q(X_i)]/2$ for each i. then $\sum_{i=1}^{n} \alpha_i[X_i(t) - y_i] < -\epsilon$ for all t even though y_i is on the proper side of $P(X_i)$ for all i. Hence the upper and/or lower previsions are incoherent by the definition of Section 2.

Suppose that a set $\{P(X) : X \in \mathcal{X}\}$ of upper and/or lower previsions is incoherent by the definition of Section 2. Let α_i have the proper signs and let y_i be on the proper side of $P(X_i)$ for each i while $\sum_{i=1}^{n} \alpha_i[X_i(t) - y_i] < -\epsilon$ for all t. Let $Q(X_i) = 2y_i - P(X_i)$ and $a_i = \alpha_i/(2[P(X_i) - Q(X_i)])$. Then $a_i > 0$ for all i and (7.2) tells us that (7.1) holds. This makes the upper and/or lower previsions incoherent by the definition of this section.

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