HOW SETS OF COHERENT PROBABILITIES MAY SERVE AS MODELS FOR DEGREES OF INCOHERENCE

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We introduce two indices for the degree of incoherence in a set of lower and upper previsions: maximizing the rate of loss the incoherent bookmaker experiences in a Dutch Book, or maximizing the rate of profit the gambler achieves who makes Dutch Book against the incoherent bookmaker. We report how efficient bookmaking is achieved against these two indices in the case of incoherent previsions for events on a finite partition, and for incoherent previsions that include also a simple random variable. We relate the epsilon-contamination model to efficient bookmaking in the case of the rate of profit.

Keywords: Dutch Book, coherence, ϵ-contamination model

1. Introduction

It is a familiar remark that de Finetti's *Dutch Book* argument provides a simple dichotomy between coherent and incoherent previsions.[1] For our presentation here, consider the following version of his argument, which we present as a two-person, zero-sum game between a *Bookie*, who is the subject of the argument, and a *Gambler*, who is the opponent.

Let X be a (bounded) random variable defined on some space S of possibilities. The Bookie is required to offer his/her prevision $p(X)$ on the condition that the Gambler may then choose a real quantity $\alpha_{X,p(X)}$ resulting in a payoff to the Bookie of $\alpha_{X,p(X)}[X - p(X)]$ with the opposite payoff to the Gambler - a zero-sum game. The Bookie's previsions for a set of random variables are *incoherent* if there is a (finite) selection of non-zero α's by the Gambler that results, by summing, in a (uniformly) negative payoff to the Bookie and a (uniformly) positive payoff to the Gambler. The *Bookie's* previsions are *coherent*, otherwise. This leads to de Finetti's *Dutch Book Theorem* – The Bookie's previsions are coherent if and only if they are the expectations of a (finitely additive) probability distribution.

de Finetti extends his analysis to include assessments of conditional previsions, given an event F, through called-off wagers using the indicator for F, χ_F, of the form $\alpha_{X,p(X),F}\chi_F[X - p(X)]$.

Moreover, when the random variables X are restricted to indicator functions for events, E, the Bookie's previsions are coherent if and only if they are the conditional probabilities of a single (finitely additive) probability. In this case, the magnitude $\mid \alpha_{E,p(E),F} \mid$ is the stake for each wager, and the sign of $\alpha_{E,p(E),F}$, positive or negative, determines whether the Bookie bets respectively, on or against E, called-off if F fails to occur.

It is a familiar concern that de Finetti's criterion of coherence requires that the Bookie posts a *single* prevision, or called-off prevision given F, for each X. For betting on events, this amounts to stating his/her "fair (called-off) odds": odds that the Gambler may use regardless the sign of the coefficient α. In response to this concern, the game has been relaxed to permit what Smith called lower and upper odds.[2] That is, in the case of indicator variables, the Bookie may post one prevision p – a "lower" probability used with positive α – for wagering on E, and another prevision q – an "upper" probability used with negative α – for wagering against E. In effect, the Bookie asserts that at odds of $p : 1 - p$ or less he/she will bet on E, whereas at odds of $q : 1 - q$ or greater he/she will bet against E.

de Finetti's Dutch Book theorem generalizes in this setting to assert, roughly, that the Bookie's lower and upper previsions are coherent if and only if they are, respectively, the lower and upper expectations of a convex set of (finitely additive) probability distributions. (We give a precise statement of this result.[3]) This generalization, however, retains the initial dichotomy: the Bookie's previsions are coherent or else the Gambler can make a Dutch Book and achieve a sure return.

2. Degrees of Incoherence

We introduce two indices of incoherence: a *rate of loss* for the Bookie and a *rate of profit* for the Gambler.[4] These index the amount of the Gambler's sure-gain against either of two "escrow" accounts, accounts that reflect the portion of the total stake each player contributes. The rate of loss indexes the Gambler's guaranteed sure gain (i.e., the minimum of the Bookie's assured loss) against the proportion of the total stake contributed by the Bookie. The rate of profit indexes the Gambler's guaranteed sure gain against his/her own contribution to the total stake.

Formally, these two indices of incoherence are defined as follows. For convenience, we denote by "t" the states on which a random variable is defined. For each gamble Y that is acceptable to the Bookie, $Y = \alpha_{X,p(X)}(X - p(X))$, the quantity $e_B(Y) = \sup_t -Y(t)$ is what we call the *Bookie's escrow* for that wager. It is the maximum loss to the Gambler that the Bookie might incur from that one transaction. Likewise, the quantity $e_G(Y) = \sup_t Y(t)$ is what we call the *Gambler's escrow* for that wager. It is the maximum loss to the Bookie that the Gambler might incur from that one transaction.

When a finite set of wagers, $\{Y_i = \alpha_{X_i,p(X_i)}(X_i - p(X_i)) : i = 1, \ldots, n\}$, is contracted between the Bookie and Gambler, we use the sum of their respective escrows from the n individual wagers to create the two (respective) *composite escrows*. That is, we imagine that each wager must be secured on its own. Denote these composite escrows, respectively, by $E_B = \sum_{i=1}^{n} e_B(Y_i)$ and $E_G = \sum_{i=1}^{n} e_G(Y_i)$, for the Bookie and the Gambler. A player's composite escrow, thus, is the total of the maximum possible losses that the player faces, reflecting each transaction on its own merits. (In another work, we characterize alternative escrows systems for standardizing the degree of incoherence in a set of wagers.[5])

Next, define the guaranteed loss L to the Bookie, associated with these n wagers, by the quantity

$$L(\{Y_i : i = 1, \ldots, n\}) = -\min\left\{0, \sup_t -\sum_{i=1}^{n} Y_i(t)\right\},$$

which is the guaranteed gain to the Gambler from these n transactions. Thus, precisely when the Gambler makes a Dutch Book against the Bookie, L is positive. Last, define two *rates of incoherence* for these n wagers as follows:

$L(\{Y_i : i = 1, \ldots, n\})/E_B$,	the *rate of guaranteed loss* to the Bookie, and
$L(\{Y_i : i = 1, \ldots, n\})/E_G$,	the *rate of guaranteed profit* to the Gambler.

The rate of loss indexes the proportion of the Bookie's resources, as measured by his escrow, that the Gambler can be certain of winning with this combination of n wagers. On the other hand, the rate of profit indexes the Gambler's sure gain from the Bookie against the total resources he or she must be prepared to escrow when covering these n wagers. To repeat, we imagine that each wager must be secured on its own. For example, if the Gambler has several alternatives for making different Books, against one or several Bookies, the rates of profit from these alternatives inform the Gambler how to guarantee the maximum winnings, standardized by the magnitude of the escrows that he or she must commit to securing these transactions.

In what follows, we focus on the second of these two indices: the rate of profit achieved by the Gambler. Of course, there are more than these two ways of formalizing degrees of incoherence. Nau gives a flexible framework that incorporates our "rate of loss" as a special case, for example.[6]

2.1. *Incoherence for events in a partition*

Let $\{A_i : i = 1, \ldots n\}$ be a partition of the sure-event by n non-empty events, with $n > 1$. Let $0 \leq p_i \leq q_i \leq 1$ be the Bookie's lower and upper previsions for the $A_i (i = 1, \ldots, n)$. Order the A_i so that $p_i \leq p_j$ for $i \leq j$. Let $s^+ = \sum q_i$ and let $s^- = \sum p_i$, so that the Bookie is incoherent if either $s^+ < 1$ or $1 < s^-$.

Theorem 2.1[4]

(i) If $s^+ < 1$ then the maximum rate of guaranteed profit equals $(1 - s^+)/s^+$ and is achieved when the Gambler sets all the $\alpha_i = -1/s^+$.

(ii) If $s^- > 1$, then the Gambler maximizes the minimum rate of profit by choosing the stakes according to the following rule: Let k^* be the first k such that $\sum_{i=n-k+1}^{n} p_i > 1 + (k-1)p_{n-k}$, with $k^* = n$ if this equality always fails.

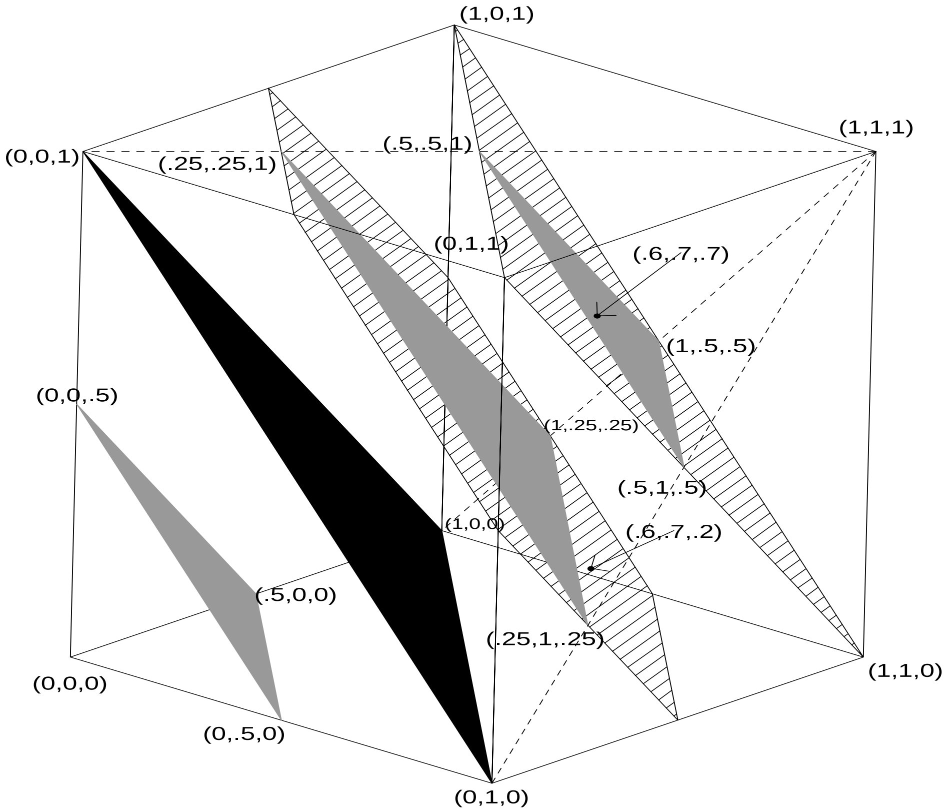

Figure 1: Sets of bookie's previsions for which the gambler uses all three events (grey shading) or just two events (line shading). The coherent simplex is shaded black. The broken lines give the edges of the triangular cone, the projection of the coherent simplex.

> Then the Gambler sets α_i all equal and positive for $i \geq n - k^* + 1$, and sets $\alpha_i = 0$ for all $i < n - k^*$.

Figure 1 illustrates this result for the case with $n = 3$ atoms, a ternary partition. The set of coherent de Finetti-previsions is represented by the black trianglular hyperplane: the simplex with extreme values (1,0,0), (0,1,0), (0,0,1). The set of incoherent "lower' probabilities, where $s^- > 1$, lies above it. The selected hyperplanes in the figure are comprised of lower probabilities with $s^- = 1.5$ and $s^- = 2.0$. For those lower previsions in the line-shaded region, outside the projection of the coherent simplex, the Gambler maximizes his/her rate of profit by ignoring the Bookie's prevision on A_3, and achieving book by having the Bookie bet on each of A_1 and A_2, at equal stakes. That would be the case if the Bookie's lower previsions were, e.g., (.6, .7, .2) when the rate of profit is 3/7. If, however, the Bookie's previsions were inside the projection of the coherent simplex, e.g., (.5, .5, .5), then the Gambler's rate of profit is only 1/3, achieved with equal stakes on each of the three atoms.

2.2. *Incoherence with previsions for a simple random variable*

Next, consider the addition of a single random variable X defined by a (finite) partition, $\{A_i : i = 1, \dots n\}$, as in the subsection above. Let $X\chi_{A_i} = x_i$ and order these values so that they are increasing, $x_i \leq x_j$ for $i \leq j$. For the next result, we assume that the Bookie gives n previsions for the events A_i, $p_i = p(A_i)$, which are singly coherent, $0 \leq p_i \leq 1$. Also, the Bookie gives a prevision p_X for X, where we assume that prevision is coherent by itself, i.e., $x_1 \leq p_X \leq x_n$. For simplicity, we state the following result for the case $s = \sum_{i=1}^{n} p_i \leq 1$.

Define these seven quantities,

$$
\begin{aligned}
s &= \sum_i p_i, \\
\mu &= \sum_i x_i p_i, \\
\delta &= p_X - \mu., \\
s_{-,k} &= \sum_{i=1}^{k} p_i, \\
s_{+,k} &= \textstyle\sum_{i=n-k+1}^{n} p_i, \\
v_-(k) &= p_x - \sum_{i=1}^{k} x_i p_i - (1 - s_{-,k})x_1, \text{ and} \\
v_+(k) &= (1 - s_{+,k})x_n + \sum_{i=n-k+1}^{n} x_i p_i - p_x.
\end{aligned}
$$

Theorem 2.2[4]

The Gambler achieves the maximum guaranteed rate of profit, as follows:

(i) If $\delta < (1 - s)x_1$, let k^* be the smallest value of k such that $v_-(k) < 0$. Then set $\alpha_X = -1$, set $\alpha_i = x_i - x_{k^*}$ for $i \leq k^*$ and set $\alpha_i = 0$ for $i > k^*$.

(ii) If $\delta > (1 - s)x_n$, let k^* be the smallest value of k such that $v_+(k) < 0$. Then set $\alpha_X = 1$, set $\alpha_i = -x_i + x_{n-k^*}$ for $i \geq n - k^*$ and $\alpha_i = 0$ for $i < n - k^*$.

(iii) If $(1-s)x_1 \leq \delta \leq (1-s)x_n$, then set $\alpha_X = 0$ and apply the previous theorem, i.e., ignore the Bookie's prevision for X but, instead, use solely the incoherence among the p_i.

A Corollary to this Theorem is interesting and intelligible on its own. Having already given the (possibly incoherent) previsions p_i, and now obliged to provide the additional prevision p_X, the Bookie can ask how to avoid increasing the rate of profit that the Gambler may achieve.

Corollary 2.1.
The Gambler's rate of profit after learning the Bookie's prevision p_X does not increase if and only if p_X satisfies:

$$\mu + (1-s)x_1 \leq p_X \leq \mu + (1-s)x_n. \tag{1}$$

That is, the corollary identifies the Bookie's minimax strategies for augmenting the previsions p_i for the events A_i, with a single new prevision p_X for X.

This corollary applies to "called-off" betting as a special case:

Consider the ternary partition and random variable X whose values are given in the second row of the following table.

	A_1	A_2	A_3
	0	p_X	1
Thus	$-\alpha p(X)$	0	$\alpha[1-p(X)]$

are the three corresponding payoffs to the Bookie associated with the wager $\alpha\chi_{A_2^C}[X - p(X)]$, a called off wager on A_3 versus A_1 given $A_1 \cup A_3$. Then, e.g., with $s \leq 1$, having already announced the previsions $p_i (i = 1,2,3)$, the Bookie's minimax strategies for restraining the Gambler's rate-of-profit satisfies: $p_2 p_X + p_3 \leq p_X \leq p_2 p_X + p_3 + 1 - s$.

It is interesting to note that choosing the pseudo-Bayes' 'conditional' value $p_X = p_3/(p_1+p_3)$ always satisfies these inequalities. In other words, the incoherent Bookie can take advantage of the fact that the pseudo-Bayes' solution is minimax. You don't have to be coherent to like Bayes' solutions! Of course, if $s = 1$, so that the Bookie is coherent, the sole minimax solution is just the Bayes' solution.

3. Epsilon-contamination and the rate of guaranteed profit

The Gambler's decisions in the first of the two Theorems, in section 2.1 above, can be explained with an ϵ-contamination model, through the Bayesian "dual" to the minimax strategies for this case. For the Gambler to accept wagers when the Bookie offers "upper" probabilities, the Gambler must find these wagers acceptable as "lower" probabilities in a rational decision. Similarly, for the Gambler to accept wagers when the Bookie offers "lower" probabilities, the Gambler must find these wagers acceptable as "upper" probabilities in a rational decision.

Given a fixed probability distribution, p^* and $\epsilon, 1 \geq \epsilon \geq 0$, the ϵ-contamination model of probabilities is the set of probabilities $M_{p^*} = \{(1-\epsilon)p^* + \epsilon q$: with q an arbitrary probability$\}$. Equivalently for finite algebras, an ϵ-contaminated model is given by specifying a coherent set of "lower" probabilities for the atoms of the algebra.

When the Bookie is incoherent with "upper" probabilities, $s^+ < 1$, by the fact just stated, these values may be the coherent lower probabilities for the Gambler using an ϵ-contamination model. That is, for $\epsilon > 0$, set $p_i = (1-\epsilon)p_i^*$, so that $s = 1-\epsilon$. In fact, the Gambler maximizes his/her expected rate of profit according to this (convex) set by wagering as indicated in (i) of the Theorem 2.1.

However, when the Bookie is incoherent with "lower" probabilities, $s^- > 1$, it is not always the case that these can be the coherent "upper" probabilities for an ϵ-contamination model. Precisely when the Bookie's "lower" probabilities fall within the projection of the coherent simplex, when they fall within the triangular region illustrated in the Figure, then the Gambler may use these as the coherent "upper" probabilities from an ϵ-contamination model. Otherwise, the Gambler fits the "largest" ϵ-contamination model that is allowed by the Bookie's offers. Expressed in other words, the strategies reported by the Theorem are those which give the Gambler a positive expected value for each component wager used to make the Dutch Book, and these relate to an ϵ-contamination model, as just explained. More precisely,

Lemma 3.1.

The Bookie's incoherent "lower" probabilities p_i are a coherent set of "upper" probabilities from the ϵ-contamination model with $\epsilon = (s-1)/(n-1)$ if and only if $p_1 \geq \epsilon$.

Proof. By assumption that no one of the Bookie's previsions alone is incoherent, we have $n \geq 2$. Then, with $s = \sum_i p_i$, the necessary and sufficient for an ϵ-contamination model to fit a coherent set of (ordered) upper probabilities p_i is that $p_i = (1-\epsilon)p_i^* + \epsilon$. Substituting $\epsilon = (s-1)/(n-1)$ identifies the quantities $p_i^* = [(n-1)p_i - (s-1)]/(n-s)$, which constitute a probability distribution if and only if $p_1 \geq (s-1)/(n-1)$. □

We obtain the threshold quantity, k^*, of Theorem 2.1(ii), by noting that the inequality $\sum_{i=n-k+1}^{n} p_i \leq 1 + (k-1)p_{n-k}$ corresponds to finding the largest ϵ-contamination model that can be formed by deleting the Bookie's smallest $n-k^*$ previsions p_i and forming the quotient algebra for the remaining k^* atoms. Then, the Bookie's remaining k^* (incoherent) previsions satisfy the conditions of lemma 3.1 for an ϵ-contamination model on these k^* atoms.

Parallel analysis applies to Theorem 2.2, in section 2.2 above. This case involves the Gambler's rate of guaranteed profit when the Bookie's previsions include a set of bets on a finite partition and a prevision for one (simple) random variable defined on that partition. The inequalities of Theorem 2.2 correspond, in precisely the same way, to the upper and lower expectations from an ϵ-contamination model, based on the Bookie's incoherent "upper" previsions, i.e., when $s < 1$. In particular, the Bookie's (ordered) "upper" probabilities p_i are incoherent with $\sum_i p_i = s < 1$. These are the Gambler's coherent "lower" probabilities from an ϵ-contamination model with $\epsilon = 1 - s$ and $p_i^* = p_i/(1-s)$.

Lemma 3.2.

Let X be a random variable defined on the partition of these atoms, with ordered values $X_{(i)}$ on the atoms $A_{(i)}$ corresponding to the an ϵ-contamination model with "lower" probabilities p_i:

(i) the "lower" expectation for X is $\mu + (1-s)x_{(1)}$ and

(ii) the "upper" expectation for X is $\mu + (1-s)x_{(n)}$.

Proof. For (i) note that the "lower" expectation from the ϵ-contamination model $\underline{E}_q(\cdot)$ is given by that distribution q with $q(x_{(1)}) = (1-\epsilon)p^*(x_{(1)})+\epsilon = p(x_{(1)})+\epsilon$ and $q(x_{(i)}) = (1-\epsilon)p^*(x_{(i)}) = p(x_{(i)})$ for $i > 1$. Hence $\underline{E}_q(X) = \sum_i x_{(i)}p(x_{(i)}) + \epsilon x_{(1)} = \mu + (1-s)x_{(1)}$. For (ii), note that the "upper" expectation from the ϵ-contamination model $\overline{E}_q(\cdot)$ is given by that distribution q with $q(x_{(n)}) = (1-\epsilon)p^*(x_{(n)}) + \epsilon = p(x_{(n)}) + \epsilon$ and $q(x_{(i)}) = (1-\epsilon)p^*(x_{(i)}) = p(x_{(i)})$ for $i < n$. Hence $\overline{E}_q(X) = \sum_i x_{(i)}p(x_{(i)}) + \epsilon x_{(n)} = \mu + (1-s)x_{(n)}$. □

Thus, Corollary 2.1 asserts that the Bookie can avoid increasing the Gambler's guaranteed rate of profit by choosing a prevision for X that lies between the Gambler's "lower" and "upper" expected values for X.

4. Conclusion

This paper introduces the use of a convex set of coherent probabilities, the ϵ-contamination model, as the Bayes' dual solutions to a Gambler's maximin strategies for what we call the guaranteed rate of profit in making efficient Dutch Book against an incoherent Bookie. The two cases discussed here include (1) incoherent upper and lower previsions for events in a finite partition, and (2) a context where the Bookie includes a prevision for a simple random variable defined on this same partition.

We are studying the corresponding convex set of probabilities that are dual to the Gambler's maximin strategies for maximizing the Bookie's guaranteed rate of loss in each of these two cases. These sets involve fixing both upper and lower probability bounds on the atoms of the finite algebra, rather than merely fixing the lower probabilities, as is done in an ϵ-contamination model. We hope to report on these finding in a future publication.

Acknowledgments

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