A Compensator for Unknown Slowly Time-Varying Input Offsets in Continuous-Time State-Feedback Systems[†]

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Abstract

We consider a class of state-feedback control systems in which there are unmodelled slowly time-varying offsets in the control action. We introduce a fast-sampling compensator which can be incorporated into the control system. We show that for sufficiently high sampling rates, this compensator virtually eliminates the effect of the unknown offsets. Performance of the compensator is illustrated in a simulation.

Keywords: control, time-varying systems, state-feedback, regulation

1 Introduction

Consider the system

$$\dot{x}(t) = f(x(t)) + B[u(t) + d(t)], \tag{1}$$

where $t \geq 0$ represents time, $x(t) \in \mathbb{R}^p$, $f : \mathbb{R}^p \to \mathbb{R}^p$ is a nonlinear function satisfying the Lipschitz condition

$$\|f(x_2) - f(x_1)\| \leq \beta \|x_2 - x_1\| \quad \text{for all } x_1, x_2 \in \mathbb{R}^p, \tag{2}$$

$u(t) \in \mathbb{R}^q$, $d(t) \in \mathbb{R}^q$ represents an unknown "input offset disturbance", and B is a $p \times q$ matrix of real numbers. The initial state $x(0)$ is assumed to be finite and the process $\{u(t),\ t \geq 0\}$ is the control variable. In this paper, our goal will be to regulate the system, that is, to drive the system state to the origin and hold it there. We assume that full state measurements are available.

Let $u^* : \mathbb{R}^p \to \mathbb{R}^q$ be a state-feedback law chosen so that for all $x \in \mathbb{R}^p$,

$$f(x) + Bu^*(x) = Hx, \tag{3}$$

where the $p \times p$ matrix H has eigenvalues whose real parts are strictly negative. (We assume that the system is controllable, so that such a control law exists.)

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Now suppose that $d(t) = 0$ for all $t \geq 0$. Then the closed loop system obtained by substituting the control law

$$u(t) = u^*(x(t)) \tag{4}$$

into (1) is simply $\dot{x}(t) = Hx(t)$, which is globally asymptotically stable. That is to say, from every starting point $x_0 \in \mathbb{R}^p$, $\lim_{t\to\infty} x(t) = 0$. The requirement that H has eigenvalues strictly in the left half-plane forms the basis for many control design techniques.

It is of practical concern, however, to consider the situation where $d(t)$ is *not* necessarily equal to zero. This happens, for instance, when there is an uncontrollable time-varying offset in the control action. In a physical system, this could be due to temperature changes in the equipment, or a progressive failure in one or more components of the system.

Consider the simplest of these cases, when $d(t)$ is equal to a non-zero constant d (not depending on t). In this case, the closed-loop system under the control law (4) is

$$\dot{x}(t) = Hx(t) + Bd.$$

Although this system is stable in the sense that $\lim_{t\to\infty} < \infty$, it is generally not the case that $\lim_{t\to\infty} = 0$. The dynamical system has a globally asymptotically stable point at a point which is not the origin. This kind of behaviour is clearly undesirable in practice.

In this paper, we investigate the performance of a particular type of control law which virtually eliminates the undesirable effects caused by the presence of an unknown slowly time-varying $d(t)$. We make the assumption that

$$\|d(t) - d(s)\| \leq \gamma|t - s|, \quad \forall s, t \geq 0, \tag{5}$$

where $\|\cdot\|$ denotes the Euclidean norm, and $d(0) = 0$. We choose the control law to be

$$u(t) = u^*(x(t)) + u_c(t), \tag{6}$$

where $u_c(t)$ can be regarded as an additional "compensating control". The expression for $u_c(\cdot)$ is very simple, and is easily computed. In addition, by virtue of the structure of (6), it is possible to gain the benefits of the use of this control law simply by adding a module to an existing linear state-feedback control system. The added module simply has to compute $u_c(t)$ and add it to the signal $u(t)$ of the already-existing controller.

In the next section we propose a particular piecewise-constant law for $u_c(t)$. This law is based on previous control laws such as those discussed in [5], [6] [1], and [2]. We choose a piecewise constant control law because this models the situation where a control scheme is being implemented with A/D and D/A converters and some kind of digital computing device. In Section 3, we analyze the closed-loop performance under the proposed control. For the sake of readability, some proofs of results in this section are relegated to an appendix. In Section 4, we simulate the behaviour of a controlled system with and without the compensator.

2 The Control Law

Before introducing our proposed control law, we need the following notation. Let $\delta > 0$ be some fixed sampling interval. For $k = 0, 1, 2, \ldots$, we define

$$x_k = x(k\delta), d_k = d(k\delta) \text{ and } u_k = u(k\delta). \tag{7}$$

Clearly, if we could set $u_c(t) = -d(t)$, then our objective would be achieved. The closed-loop system would be $\dot{x}(t) = Hx(t)$, which is globally asymptotically stable. Unfortunately, this is not possible. The disturbance $d(t)$ is assumed to be unknown. It is possible, however, to come close to this.

To see how this can be done, consider the approximating system

$$x^a_{k+1} = x^a_k + \delta f(x^a_k) + \delta B(u_k + d_k), \tag{8}$$

with $x_0 = x(0) = x_{(0)}$. This system is a discrete-time Euler approximation to the continuous-time system (1).

Now let P be any matrix such that

$$PB = I_{q\times q}, \tag{9}$$

the $q\times q$ identity matrix. (As long as B is of rank q, then such a matrix can be found.) Equation (8) can be written as

$$\delta B d_{k-1} = x^a_k - x^a_{k-1} - \delta f(x^a_{k-1}) - \delta B u_{k-1}. \tag{10}$$

Premultiplying both sides by P and dividing by δ gives us

$$d_{k-1} = P\delta^{-1}(x^a_k - x^a_{k-1}) - Pf(x^a_{k-1}) - u_{k-1}. \tag{11}$$

At time $k\delta$, it is possible to compute d_{k-1} using this expression, but not to compute d_k; in order to compute d_k we would need to now the future state x_{k+1}. However, if d_k is not varying too rapidly, then d_k will be reasonably close to d_{k-1}, so to obtain our control law, we simply put $u_c(t) = -d_{k-1}$ for $t \in [k\delta, (k+1)\delta)$, with x^a_j replaced by $x_j = x(j\delta)$. Since this is not well defined for $k = 0$, we put $u_c(t) = 0$ for $t \in [0, \delta)$. To be more precise, we choose

$$u_c(t) = u_{c,k},\ t \in [k\delta, (k+1)\delta),\ k = 0, 1, 2, \ldots, \tag{12}$$

where

$$u_{c,k} = \begin{cases} 0, & k = 0 \\ -P\delta^{-1}(x_k - x_{k-1}) + Pf(x_{k-1}) + u_{k-1}, & k = 1, 2, \ldots. \end{cases} \tag{13}$$

3 Performance Analysis

To analyze the closed-loop performance of the system, we begin by constructing a difference equation for the sampled system. It is not difficult to verify that the solution to (1) satisfies

$$x_{k+1} = x_k + \delta f(x_k) + \delta B(u_k + d_k) + \xi_{k+1} \tag{14}$$

for $k = 0, 1, 2, \ldots$, where

$$\xi_{k+1} = \int_0^\delta \{f(x(k\delta + s)) - f(x_k) + B[u(k\delta + s) - u_k + d(k\delta + s) - d_k]\}ds. \tag{15}$$

(Recall the definitions (7).)

The closed-loop equation for the sampled system (obtained by substituting (6) and (12) into (14)) is then

$$\begin{aligned} x_{k+1} &= x_k + \delta f(x_k) + \delta B(u_k + d_k) + \xi_{k+1} \\ &= x_k + \delta f(x_k) + \delta B(u^*(x_k) + u_{c,k} + d_k) + \xi_{k+1} \\ &= (I + \delta H)x_k + \delta B(u_{c,k} + d_k) + \xi_{k+1}. \end{aligned} \tag{16}$$

Rearranging (14) with k replaced by $k-1$, pre-multiplying by P, and comparing with (13), we can see that

$$u_{c,k} = -d_{k-1} - P\xi_k/\delta \tag{17}$$

for integers $k \geq 0$ (adopting the convention that $\xi_0 = d_{-1} = 0$). Substituting (17) into (16) gives us

$$\begin{aligned} x_{k+1} &= (I + \delta H)x_k + \delta B(d_k - d_{k-1} - P\xi_k/\delta) + \xi_{k+1} \\ &= (I + \delta H)x_k + \delta B(d_k - d_{k-1}) + \xi_{k+1} - BP\xi_k \end{aligned} \tag{18}$$

for integers $k \geq 0$. Iterating this, we obtain

$$x_{k+1} = L^{k+1}x_0 + \sum_{j=0}^{k} L^j[\delta B(d_{k-j} - d_{k-j-1})] \tag{19}$$

$$+\sum_{j=0}^{k} L^j[\xi_{k+1-j} - BP\xi_{k-j}], \tag{20}$$

where for the sake of conciseness, we define $L = I + \delta H$.

In order to further analyze closed-loop performance, we need the following results.

Lemma 3.1 *Suppose that H is a $p\times p$ matrix and let $\{\lambda_i,\ i = 1, \ldots, p\}$ denote the p (not necessarily distinct) eigenvalues of H . Suppose also that the real parts of λ_i are strictly negative. Let*

$$\delta_0(H) := \min_{i=1,\ldots,p} \frac{\Re(\lambda_i)}{|\lambda_i|^2} \tag{21}$$

and

$$\nu(H) := \min_i \frac{|\lambda_i|(|\lambda_i| - |\Im(\lambda_i)|)}{|\Re(\lambda_i)|}, \tag{22}$$

where $\Re(\lambda_i)$ and $\Im(\lambda_i)$ denote the real and imaginary parts of λ_i, respectively. Let $\rho(A)$ denote the spectral radius of the $p \times p$ matrix A. If

$$0 < \delta < \delta_0(H), \tag{23}$$

then

$$\rho(I + \delta H) \leq 1 - \delta\nu(H). \tag{24}$$

The next result, which is a generalization of a result presented in [2], establishes a bound on the spectral norm of $(I + \delta H)^k$.

Lemma 3.2 *Let H be a $p \times p$ matrix with eigenvalues whose real parts are strictly negative. Let $\delta_0(H)$ and $\nu(H)$ be given respectively by (21) and (22). For any matrix A, let $\|A\|_\epsilon$ denote its Euclidean norm, $\|A\|$ denote its spectral (or operator) norm and let A^* denote its conjugate transpose. Then if $0 \le \delta \le \delta_0(H)$,*

$$\|(I + \delta H)^k\| \le \sum_{j=0}^{\min(p-1,k)} \binom{k}{j} [1 - \delta\nu(H)]^{k-j} [\delta\mu(H)]^j, \tag{25}$$

where

$$\mu(H) = \left(\frac{p^3 - p}{12}\right)^{1/4} \sqrt{\|H^*H - HH^*\|}. \tag{26}$$

Proof: It follows from Theorems 1 and 2 of [4] that for any $p \times p$ matrix A,

$$\|A^k\| \le \sum_{j=1}^{\min(p-1,k)} \binom{k}{j} \rho(A)^{k-j} \mu(A)^j.$$

Replacing A with $I + \delta H$, observing that $\mu(I + \delta H) = \delta\mu(H)$ and making use of Lemma 3.1 to bound $\rho(I + \delta H)$, we get the desired result. □

As a direct consequence of Lemma 3.2 (summing inequality (25) for $k = 0, 1, 2, \ldots$) we have the following.

Corollary 3.3 *Let H be a $p \times p$ matrix with eigenvalues whose real parts are strictly negative. Let $\delta_0(H), \nu(H)$ and $\mu(H)$ be given respectively by (21), (22) and (26). If $0 < \delta \le \delta_0(H)$, then*

$$\sum_{j=0}^{\infty} \|(I + \delta H)^j\| \le \frac{\eta(H)}{\delta}, \tag{27}$$

where

$$\eta(H) = \sum_{k=0}^{p-1} \left[\frac{\mu(H)}{\nu(H)}\right]^k. \tag{28}$$

Now we can make use of Corollary 3.3 to establish the following results.

Lemma 3.4 *If $0 < \delta \le \delta_0$, then for all non-negative integers k,*

$$\|\sum_{j=0}^{k} (I + \delta H)^j \delta B[d_{k-j} - d_{k-j-1}]\| \le \delta\gamma\|B\|\eta(H). \tag{29}$$

Proof: This follows directly from inequality (5) and Corollary 3.3. □

Now let

$$\begin{aligned}
\phi_1(\delta) &= (\exp(\delta\|H\|) - 1 - \delta\|H\|)(\delta\|H\|)^{-1}\|BP\| && (30)\\
\phi_2(\delta) &= (\exp(\delta\|H\|) - 1 - \delta\|H\|) && (31)
\end{aligned}$$

and

$$\phi_3(\delta) = 2(\exp(\delta\|H\|) - 1 - \delta\|H\|)\gamma(\delta\|B\| + 1). \quad (32)$$

The following lemma bounds more terms on the right hand side of equation (18). A proof is given in the appendix.

Lemma 3.5 *Let $\phi_1(\delta), \phi_2(\delta)$ and $\phi_3(\delta)$ be as defined by (30), (31) and (32), respectively. Suppose that $0 < \delta \le \delta_0(H)$ and that $|\phi_1(\delta)| < 1$. Then*

$$\|\sum_{j=0}^{k} L^j BP\xi_{k-j}\| \le \|BP\| \left(\sum_{j=0}^{k} c_j\|x_{k-j}\| + \phi_4(\delta) \right)$$

and

$$\|\sum_{j=0}^{k} L^j \xi_{k+1-j}\| \le \phi_2(\delta)\|x_{k+1}\| + \sum_{j=0}^{k-1} c_{j+1}\|x_{k-j}\| + \phi_4(\delta),$$

where

$$c_j = \phi_2(\delta) \sum_{m=0}^{j} \|L^m\|\phi_1(\delta)^{j-m}$$

and

$$\phi_4(\delta) = \frac{\phi_3(\delta)\nu(H)}{\delta[1 - \phi_1(\delta)]}. \quad (33)$$

Before we can directly analyze equation (16), we will also need the following variation of the Bellman-Gronwall Lemma. (A proof can be found in [3].)

Lemma 3.6 *Suppose that $\{b_k,\ k = 0, 1, 2, \ldots\}$, $\{a_k,\ k = 0, 1, 2, \ldots\}$ and $\{g_k,\ k = 0, 1, 2, \ldots\}$ are non-negative real-valued sequences, and that*

$$b_{k+1} \le \sum_{j=0}^{k} a_j b_{k-j} + g_{k+1} \quad (34)$$

with $\sum_{j=0}^{\infty} a_j = 1 - \epsilon < 1$. If $\{g_k\}$ is a sequence bounded above by g^*, and $\limsup_{k\to\infty} g_k \le g$, then

$$b_k \le \max(b_0, g^*/\epsilon), \quad \textit{for all } k, \quad (35)$$

and

$$\limsup_{k\to\infty} b_k \le g/\epsilon.$$

We are now in a position to state our main result.

Theorem 3.7 *Suppose that $u^*(\cdot)$ is chosen to satisfy (3) for some matrix H with eigenvalues whose real parts are strictly negative, and the control law specified by equations (6) and (13) is applied to the system (1). Let $\delta_0(H)$,$\phi_1(\delta)$,$\phi_2(\delta)$,$\phi_3(\delta)$ and $\phi_4(\delta)$ be given by (21), (30), (31), (32) and (33),respectively, and let*

$$\phi_5(\delta) = \frac{(1+\|BP\|)\phi_2(\delta)\eta(H)}{\delta[1-\phi_1(\delta)][1-\phi_2(\delta)]}. \tag{36}$$

If $0<\delta\le\delta_0(H)$, $\phi_1(\delta)<1$, $\phi_2(\delta)<1$ *and* $\phi_5(\delta)<1$, *then*

$$\limsup_{k\to\infty}\|x_k\| \le \frac{\delta\gamma\|B\|\eta(H)+(\|BP\|+1)\phi_4(\delta)}{[1-\phi_2(\delta)][1-\phi_5(\delta)]}. \tag{37}$$

Proof: Taking norms on both sides of (16), making use of the triangle inequality and applying the results of Lemmas 3.4 and 3.5, we get

$$\begin{aligned}\|x_{k+1}\| \;\le\;& \|L^{k+1}\|\|x_0\| + \delta\gamma\|B\|\eta(H) + \|BP\|\left[\sum_{j=0}^{k} c_j\|x_{k-j}\| + \phi_4(\delta)\right]\\ &+\phi_2(\delta)\|x_{k+1}\| + \sum_{j=0}^{k-1} c_{j+1}\|x_{k-j}\| + \phi_4(\delta),\end{aligned}$$

or

$$\begin{aligned}\|x_{k+1}\| \;\le\;& (1-\phi_2(\delta))^{-1}\left[\|L^{k+1}\|\|x_0\| + \sum_{j=0}^{k}(\|BP\|c_j + c_{j+1})\|x_{k-j}\|\right]\\ &+(1-\phi_2(\delta))^{-1}\left[\delta\gamma\|B\|\eta(H)+(\|BP\|+1)\phi_4(\delta)\right].\end{aligned}$$

Next, applying Lemma 3.6 with $b_k = \|x_k\|$, $a_k = (1-\phi_2(\delta))^{-1}(\|BP\|c_k + c_{k+1})$ and $g_k = (1-\phi_2(\delta))^{-1}[\|L^{k+1}\|\|x_0\| + \delta\gamma\|B\|\eta(H) + (\|BP\|+1)\phi_4(\delta)]$ and observing that

$$\begin{aligned}\sum_{j=0}^{\infty} \;\le\;& (1-\phi_2(\delta))^{-1}(\|BP\|+1)\sum_{j=0}^{\infty} c_j\\ \le\;& (1-\phi_2(\delta))^{-1}(\|BP\|+1)\phi_2(\delta)\sum_{j=0}^{\infty}\|L^j\|\sum_{k=0}^{\infty}\phi_1(\delta)^k\\ \le\;& \phi_5(\delta),\end{aligned}$$

we get the desired result. □

Remark: Since $\phi_1(\delta)=O(\delta)$, $\phi_2(\delta)=O(\delta^2)$ and (consequently) $\phi_5(\delta)=O(\delta)$, there always exists δ' such that for $0<\delta<\delta'$, the conditions of Theorem 3.7 are satisfied. In other words, by sampling

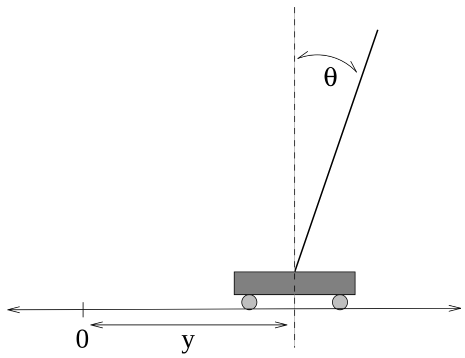

Figure 1: An Inverted Pendulum

the system at a sufficient rate, it is always possible to stabilize the system and ensure the behaviour prescribed by (37).

Remark: Since $\phi_4(\delta) = O(\delta)$, the limiting bound on the absolute deviation of the system state from the origin also approaches zero as δ approaches zero.

4 Simulations

We examine the performance of the compensator by simulating the behaviour of a controlled inverted pendulum. All simulations are carried out using Simulink version 2.2 with Matlab version 5.2.

The system is illustrated in Figure 1. The x-axis represents a "track". A cart sits on the track, and can be accelerated to the right or left by applying a voltage to its motor system. The object is, given a starting position $y(0)$ and angle $\theta(0)$, to choose a state-feedback control to drive the vector $(\theta(t), y(t))^T$ to the origin (i.e. to center the cart, with the inverted pendulum standing up straight).

The system can be linearized about the point $(0,0)$ to obtain

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{38}$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2.7530 & -10.9530 & 0.0043 \\ 0 & 28.5810 & 24.9180 & -0.0441 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1.943 \\ -4.439 \end{bmatrix}, \tag{39}$$

$x(t) = (y(t), \theta(t), \dot{y}(t), \dot{\theta}(t))^T$ and $u(t) \in \mathbb{R}$.

The state-feedback law $u^*(x(t)) = Kx(t)$ with $K = (1, 36.8586, 11.7275, 7.536)$ stabilizes the system (38).

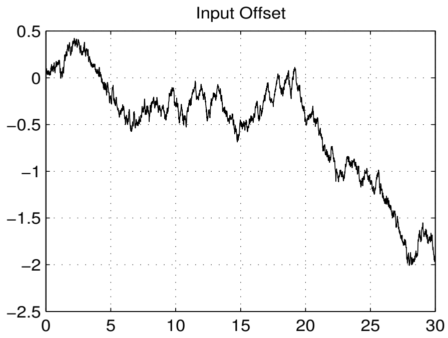

Figure 2: The input offset $d(t)$ (with time on the horizontal axis).

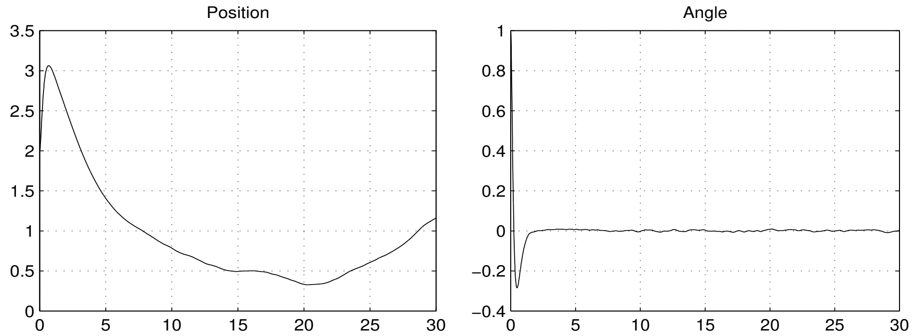

Figure 3: The position $y(t)$ and angle $\theta(t)$, without compensation, i.e. with $u(t) = u^*(x(t))$.

Now suppose that in reality, (38) does not hold exactly. Rather, we have

$$\dot{x}(t) = Ax(t) + B[u(t) + d(t)],$$

where $d(t)$ is the disturbance plotted in Figure 2. (This is one realization of an approximation to a Brownian motion process.)

If we simply apply the control $u(t) = u^*(t) = Kx(t)$, then system behaviour is as illustrated in Figure 3. Note that although the pendulum continues to stand up straight (i.e. $\theta(t) \simeq 0$ for large t), the offset $d(t)$ is causing a significant time-varying bias in the position $y(t)$.

When we apply the control with the compensator, that is, we choose $u(t) = u^*(t) + u_c(t)$, where $u_c(t)$ is given by (13) (with $\delta = 0.01$ and $P = (0, 0, 0.5147, 0)$) the state behaves as shown in Figure 5. Clearly, the time-varying bias in $y(t)$ is virtually eliminated.

5 Acknowledgements

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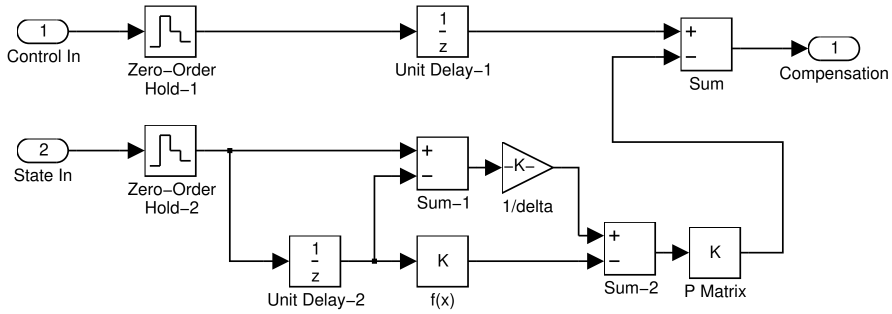

Figure 4: A Simulink representation of the compensator; The inputs (labelled “Control In” and ”State In”) are $u(t)$ and $x(t)$. The output (labelled “Compensation”) is $u_c(t)$. All zero-order-hold and unit-delay units operate with sampling rate δ. Knowledge of the function $f(\cdot)$ and of a matrix P such that $PB = I$ is required to implement the compensator.

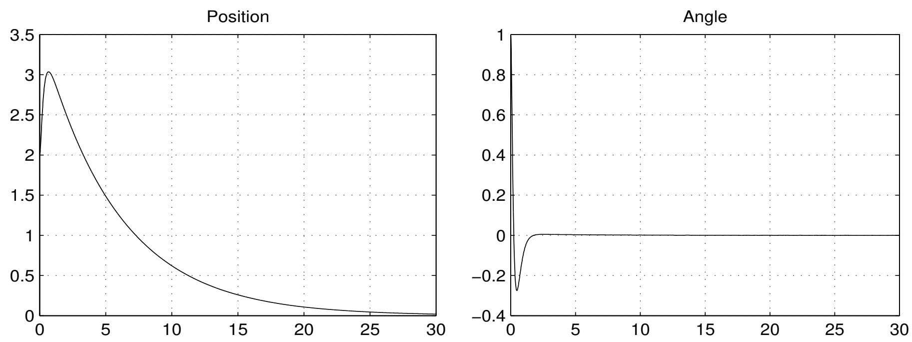

Figure 5: The position $y(t)$ and angle $\theta(t)$, with compensation, i.e. with $u(t) = u^*(x(t)) + u_c(t)$.

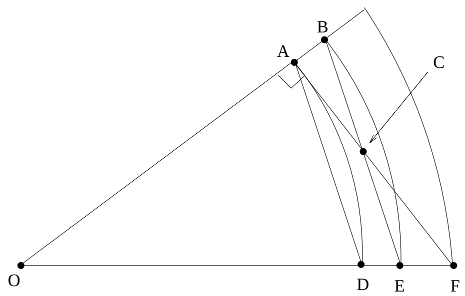

Figure 6: A geometric construction used in the proof of Lemma 3.1

6 Appendix A

This appendix contains proofs of several results stated in the main body of the paper.

Proof of Lemma 3.1: First consider a single complex value λ_i whose real part is negative. Let the points in Figure 6 represent points in the complex plane, and define them as follows: $F = 1$, O is the origin, $A = 1 + \delta_0 \lambda$, δ_0 is the unique positive value for which line segments OA and AF are perpendicular, D is the point for which $\overline{OD} = \overline{OA}$, where $\overline{XY}$ denotes the length of the line segment connecting X and Y for any points X and Y. Define ν to be the positive constant such that $\overline{OD} = 1 - \delta\nu$. Now choose any δ such that $0 \leq \delta \leq \delta_0$. Let $C = 1 + \delta\lambda$. Construct a line EB as in Figure 6 which passes through C, and is parallel to AD. Draw an arc connecting E and B such that every point on the arc is equidistant from the origin. Clearly $E = 1 - \delta\nu$. Then $|C|$ is less than or equal to the norm of any point on the arc connecting E and B, so

$$|1 + \delta\lambda_i| = |C| \leq |E| = 1 - \delta\nu. \tag{40}$$

Let $\delta_i = \Re(\lambda_i)/\left|\lambda_i\right|^2$ and $\nu_i = |\lambda_i| \left(\frac{|\lambda_i| - |\Im(\lambda_i)|}{|\Re(\lambda_i)|} \right)$. It can be shown that for λ_i, δ_i is the choice of δ_0 which ensures that line segments OA and AF are perpendicular, and that ν_i is the corresponding value of ν.

The matrix H has eigenvalues $\lambda_1, \ldots, \lambda_p$, (all of which have strictly negative real parts). Hence the matrix $I + \delta H$ must have eigenvalues $\{1 + \delta\lambda_i,\ i = 1, \ldots, p\}$. The spectral radius of $I + \delta H$ is $\max_i |1 + \delta\lambda_i|$. When $0 \leq \delta \leq \min_i \delta_i$, then from (40), $\max_i |1 + \delta\lambda_i| \leq 1 - \delta \min_i \nu_i$, which gives the desired result. □

The following lemma states a standard result in the study of discrete-time approximations to ordinary differential equations.

Lemma 6.1 *Suppose that*

$$\dot{x}(t) = a(x(t), t) \tag{41}$$

for $t \geq t_0$, where $a(\cdot, \cdot)$ satisfies the Lipschitz condition

$$\|a(y_2, s_2) - a(y_1, s_1)\| \leq \alpha_1 \|y_2 - y_1\| + \alpha_2 \|s_2 - s_1\|. \tag{42}$$

Let $x^a(t)$ be the approximation to $x(t)$ given by

$$x^a(t) = x(t_0) + (t - t_0) a(x(t_0), t_0), \quad t \geq t_0. \tag{43}$$

Then

$$\|x(t) - x^a(t)\| \leq \frac{\exp(\alpha_1(t-t_0)) - 1 - \alpha_1(t-t_0)}{\alpha_1^2}(\alpha_1\|a(x(t_0),t_0)\| + \alpha_2).$$

The next result places a preliminary bound on the magnitude of the discretization-correction terms $\{\xi_k,\ k=1,2,\ldots\}$.

Lemma 6.2 *The discretization-correction terms $\{\xi_k,\ k=1,2,\ldots\}$ defined by (15) are bounded by*

$$\|\xi_{k+1}\| \leq \phi_1(\delta)\|\xi_k\| + \phi_2(\delta)\|x_k\| + \phi_3(\delta), \tag{44}$$

where $\phi_1(\delta), \phi_2(\delta)$ and $\phi_3(\delta)$ are given by (30), (31) and (32), respectively, and we adopt the convention that $\xi_0 = 0$.

Proof: For integers $k \geq 1$ and t such that $k\delta \leq t < (k+1)\delta$, $x(t)$ satisfies (41) with

$$\begin{aligned} a(x(t),t) &= f(x(t)) + B(u(t) + d(t)) \\ &= Hx(t) + B(u_c(t) + d(t)) \\ &= Hx(t) + B(-d_{k-1} - P\xi_k/\delta + d(t)). \end{aligned}$$

The associated Lipschitz constants (c.f. (42)) are $\alpha_1 = \|H\|$ and $\alpha_2 = \gamma$. Then if $x_k^a = x(k\delta)$ and $t_0 = k\delta$, x_{k+1}^a as defined by (8) is the same as $x^a((k+1)\delta)$ as defined by (43). Applying Lemma 6.1 with $t = (k+1)\delta$, we get the desired result. For the case $k=0$, and $0 \leq t < \delta$, $x(t)$ satisfies (41) with

$$a(x(t),t) = Hx(t) + Bd(t).$$

Hence an application of Lemma 6.1 with $t_0 = 0$ and $t = \delta$ shows that inequality (44) also holds for $k=0$. □

Now we are in a position to prove Lemma 3.5.

Proof of Lemma 3.5: Iterating (44) and using the fact that $\xi_0 = 0$, we get

$$\|\xi_k\| \leq \sum_{j=0}^{k-1} \phi_1(\delta)^j[\phi_2(\delta)\|x_{k-1-j}\| + \phi_3(\delta)], \tag{45}$$

so

$$\begin{aligned} \|\sum_{j=0}^{k} L^j BP\xi_{k-j}\| &\leq \|BP\|\sum_{j=0}^{k}\|L^j\|\|\xi_{k-j}\| \\ &\leq \|BP\|\sum_{j=0}^{k}\|L^j\|\sum_{m=0}^{k-j-1}\phi_1(\delta)^m[\phi_2(\delta)\|x_{k-j-1-m}\| + \phi_3(\delta)] \\ &\leq \|BP\|\sum_{j=0}^{k} c_j\|x_{k-j}\| + \sum_{j=0}^{k}\|L^j\|\sum_{m=0}^{\infty}\phi_1(\delta)^m\phi_3(\delta) \\ &= \|BP\|\sum_{j=0}^{k} c_j\|x_{k-j}\| + \phi_4(\delta). \end{aligned}$$

A similar argument leads to the second part of the result. □

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