36-771 Fall 2018

Martingales 1 : Concentration inequalities

Proof of Ville's inequality

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1 Proof of Ville's inequality

Theorem 1 (Ville (1939)) *For any nonnegative supermartingale* (L_t) *and any* $x > 1$, *define the (possibly infinite) stopping time*

$$N := \inf\{t \geq 1 : L_t \geq x\}$$

and denote the expected overshoot when L_t *surpasses* x *as*

$$o = \mathbb{E}\left[\frac{L_N}{x}|N < \infty\right] \geq 1.$$

Then,

$$\Pr(\exists t : L_t \geq x) \leq \frac{\mathbb{E}L_0}{ox} \overset{(i)}{\leq} \frac{\mathbb{E}L_0}{x}.$$

Proof: Using the optional stopping theorem and the supermartingale convergence theorem (to establish existance of L_∞), we have the following chain of inequalities:

$$\begin{aligned}
\mathbb{E}L_0 &\overset{(ii)}{\geq} \mathbb{E}L_N \\
&= \mathbb{E}(L_N|N < \infty)P(N < \infty) + \mathbb{E}(L_\infty|N = \infty)P(N = \infty) \\
&\geq \mathbb{E}(L_N|N < \infty)P(N < \infty) \\
&= oxP(N < \infty),
\end{aligned}$$

immediately proving the theorem. ■

For nonnegative martingales, the inequality (ii) is actually an equality. For continuous-time supermartingales with continuous paths, we have $o = 1$, making inequality (i) into an equality. In fact, for continous-time martingales with continuous paths, Ville's inequality holds with equality.

References

Ville, J. (1939), *Étude Critique de la Notion de Collectif.*, Gauthier-Villars, Paris.