36-771 Fall 2018
Martingales 1 : Concentration inequalities

Universality of sub-Gamma boundaries

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Universality of sub-Gamma bounds

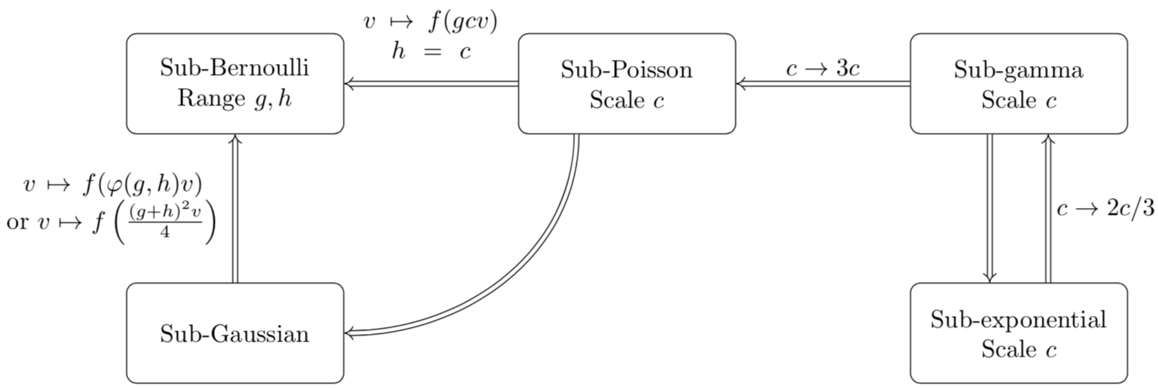

Figure 1: Schematic of relations among sub-ψ boundaries. Each arrow indicates that a sub-ψ boundary at the source node yields a sub-ψ boundary at the destination node with the modification indicated on the arrow.

A reader who is familiar with Howard et al. (2018) will note that the arrows in the above figure are reversed with respect to Figure 3 in their paper. Indeed, since any sub-Bernoulli process is also sub-Gaussian, it follows that any sub-Gaussian uniform boundary is also a sub-Bernoulli uniform boundary, and so on.

The above figure summarizes implications that hold among sub-ψ uniform boundaries. It shows, in particular, that a sub-gamma or sub-exponential uniform boundary also yields a sub-Poisson, sub-Gaussian or sub-Bernoulli uniform boundary. Indeed, sub-gamma and sub-exponential uniform bounds are universal in a certain sense:

Proposition 1 *Suppose ψ is twice continuously differentiable and $\psi(0) = \psi'(0_+) = 0$. Suppose, for each $c > 0$, $u_c(v)$ is a sub-gamma or sub-exponential uniform boundary with crossing probability α for scale c. Then $v \mapsto u_{k_1}(k_2 v)$ is a sub-ψ uniform boundary for some constants $k_1, k_2 > 0$.*

Proof: Suppose, for each $c > 0$, u_c is a sub-gamma uniform boundary for scale c. Applying Taylor's theorem to ψ at the origin, we have $\psi(x) = \left[\frac{\psi''(0_+)}{2} + h(x)\right] x^2$ where $h(x) \to 0$ as

$x \downarrow 0$. Choose $x_0 > 0$ small enough so that $\psi(x) \le \psi''(0_+)x^2$ for all $0 \le x \le x_0$. Then, setting $c = k_1 := 1/x_0$ in ψ_G, and using that fact that $\psi_G \ge \psi_N$, we have $\psi(x) \le k_2\psi_G(x)$ for all $0 \le x \le 1/c$ where $k_2 := 2\psi''(0_+)$. We conclude that, if (S_t) and (V_t) satisfy the canonical Assumption 1 for ψ, then (S_t) and (k_2V_t) satisfy Assumption 1 for ψ_G. This implies $\mathcal{P}(\exists t \ge 1 : S_t \ge u_{k_1}(k_2V_t)) \le \alpha$, which is the desired conclusion. The same argument holds if u_c is a sub-exponential uniform boundary, replacing ψ_G with ψ_E. ■

The following proposition formalizes the relationships illustrated in the above figure, and follows directly from Proposition 3 of Howard et al. (2018).

Proposition 2 *Let $u : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a sub-ψ uniform boundary with crossing probability α (we omit the dependence on $\mathbb{E}L_0$, as elsewhere).*

1. *If u is a sub-Gaussian uniform boundary, then $v \mapsto u(\varphi(g,h)v)$ is a sub-Bernoulli uniform boundary with crossing probability α for range parameters g, h, where*

$$\varphi(g,h) := \begin{cases} \frac{h^2-g^2}{2\log(h/g)}, & g < h \\ gh, & g \ge h. \end{cases} \tag{3}$$

2. *If u is a sub-Gaussian uniform boundary, then $v \mapsto u((g+h)^2v/4)$ is a sub-Bernoulli uniform boundary with crossing probability α for range parameters g, h.*

3. *If u is a sub-Poisson uniform boundary for scale c, then $v \mapsto u(gcv)$ is a sub-Bernoulli uniform boundary with crossing probability α for range parameters g, c.*

4. *If u is a sub-Poisson uniform boundary for scale c, then it is also a sub-Gaussian uniform boundary with crossing probability α.*

5. *If u is a sub-gamma uniform boundary for scale c, then it is also a sub-Poisson uniform boundary with crossing probability α for scale $3c$.*

6. *If u is a sub-gamma uniform boundary for scale c, then it is also a sub-exponential uniform boundary with crossing probability α for scale c.*

7. *If u is a sub-exponential uniform boundary for scale c, then it is also a sub-gamma uniform boundary with crossing probability α for scale $2c/3$.*

References

Howard, S. R., Ramdas, A., McAuliffe, J. & Sekhon, J. (2018), 'Exponential line-crossing inequalities', *arXiv:1808.03204 [math]* .