Martingales, Ville and Doob

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1 Martingales

Recall that a filtration $(\mathcal{F}_n)$ is a sequence of increasing sigma-fields.

Definition 1 *A sequence (S_n) is "adapted" to the filtration $(\mathcal{F}_n)$, if $S_n \in \mathcal{F}_n$ for all n. A sequence (S_n) is "predictable" (with respect to the filtration $(\mathcal{F}_n)$) if $S_n \in \mathcal{F}_{n-1}$ for all n.*

If (S_n) is integrable and adapted to $(\mathcal{F}_n)$, and it also satisfies $\mathbb{E}(S_{n+1}|\mathcal{F}_n) = S_n$, then (S_n) is called a martingale (with respect to $(\mathcal{F}_n)$). If the equality above is replaced by $\leq$, it is called a supermartingale, and if replaced by $\geq$, it is a submartingale. If $n > m$, then we have $\mathbb{E}(S_n|\mathcal{F}_m) = \mathbb{E}(S_m)$ for martingales (and $\leq$ or $\geq$ for super-/sub-martingales).

If (S_n) is a martingale wrt $(\mathcal{G}_n)$, and $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$, then we must have $\mathcal{G}_n \supset \mathcal{F}_n$ for all n and that (S_n) is a martingale wrt $(\mathcal{F}_n)$ as well. Further, $(\mathcal{F}_n)$ is the smallest filtration wrt which (S_n) is adapted, and if the filtration is not mentioned, it is understood to be $\sigma(S_1, \ldots, S_n)$.

For example, the simple random walk (S_n) is a martingale that is adapted to the "canonical" filtration $\sigma(S_1, \ldots, S_n) = \sigma(X_1, \ldots, X_n)$.

- If (S_n) is a martingale, ϕ is a convex function, and $(\phi(S_n))$ is integrable, then it is a submartingale (by Jensen's inequality).
- If (S_n) is a supermartingale and N is a stopping time, then $(S_{N\wedge n})$ is a supermartingale.

Just as it is well known that a monotonically nondecreasing sequence of real numbers with upper bound a number M converges to a limit which does not exceed M, we have the following stochastic analogue.

Theorem 1 (Martingale convergence theorem, Durrett Thm 5.2.8) *If (S_n) is a martingale with $\sup_n \mathbb{E}S_n^+ < \infty$, then S_n converges almost surely to an integrable limit X. As a corollary, if (S_n) is a positive supermartingale, then S_n converges to a limit X with $\mathbb{E}X \leq \mathbb{E}S_0$.*

Any submartingale can also be decomposed into a martingale and a predictable increasing component.

Theorem 2 (Doob's decomposition theorem, Durrett Thm 5.2.10) *Any submartingale (S_n) can be uniquely decomposed as $S_n = M_n + A_n$ where (M_n) is a martingale, and (A_n) is a predictable increasing sequence.*

There are variants of the above called Riesz's and Krickenberg's decompositions. Just as one can construct a filtration from a martingale, one can also do the reverse.

Theorem 3 (Constructing a martingale from a filtration) *Let Z be integrable, $(\mathcal{F}_n)$ be a filtration, and define $M_n = \mathbb{E}(Z|\mathcal{F}_n)$. Then (M_n) is a martingale (and moreover, it is a uniformly integrable martingale).*

This technique often allows us to use martingale methods even when there is no obvious martingale in plain sight, since one can construct it out of thin air.

2 Ville's and Doob's inequalities

The first of Doob's inequalities can be seen as a uniform generalization of Markov's inequality to submartingales.

Theorem 4 (Doob's maximal inequality for submartingales, Durrett Thm 5.4.2) *If (S_n) is a submartingale, then for any $x > 0$, we have*

$$P(\max_{1\le n\le N} S_n^+ \ge x) \le \frac{\mathbb{E}(S_N^+)}{x}$$

If $S_n = \sum_{i=1}^n X_i$ is a random walk with $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = \sigma_i^2 < \infty$, then using the fact that (S_n) is a martingale implies (S_n^2) is a submartingale, we get Kolmogorov's maximal inequality, which can be interpreted as a uniform generalization of Chebyshev's inequality to martingales. Denoting $s_n^2 := \mathrm{Var}(S_n) = \sum_{i=1}^n \sigma_i^2$, we have

$$P(\max_{1\le n\le N} S_n \ge x) \le \frac{s_N^2}{x^2}.$$

For random walks like above, the process $(S_n^2 - s_n^2)$ is also a martingale (confirm for yourself). Using this fact, if we additionally had $|X_i| \le K$, then one may also prove that

$$P(\max_{1\le n\le N} S_n \le x) \le \frac{(x+K)^2}{s_N^2}.$$

Further, for *any* zero-mean, finite-variance martingale (S_n), we have a uniform version of Upensky's inequality:

$$P(\max_{1\le n\le N} S_n \ge x) \le \frac{\mathrm{Var}(S_N)}{\mathrm{Var}(S_n) + x^2}$$

Ville's supermartingale maximal inequality is closely related to Doob's:

Theorem 5 (Ville's maximal inequality for supermartingales) *If (S_n) is a nonnegative supermartingale, then for any $x > 0$, we have*

$$P(\sup_{n \in \mathbb{N}} S_n > x) \leq \frac{\mathbb{E}S_0}{x}.$$

3 Optional Stopping (also called Optional Sampling, different from *Optimal* Stopping)

Theorem 6 (Supermartingale optional stopping, Durrett Thm 5.7.6) *If (S_n) is a nonnegative supermartingale, then for any stopping time $N \leq \infty$, we have*

$$\mathbb{E}S_N \leq \mathbb{E}S_0,$$

recalling that $S_\infty = \lim_n S_n$ exists via the martingale convergence theorem.

For martingales, equality does not hold above, because the above theorem permits unbounded stopping times. As an example, consider the simple random walk, and the stopping time $N = \inf n : S_n = 1$. Obviously $\mathbb{E}S_n = 0$ for any fixed n, but $\mathbb{E}S_N = 1$ by definition. The problem again is that N is unbounded, and indeed $\mathbb{E}N = \infty$. Instead, we have the following:

Theorem 7 (Doob's martingale optional sampling, Gut Corollary 7.1) *If (S_n) is a martingale, and N is a bounded stopping time, i.e. $P(N \leq K) = 1$ for some constant K, then $\{S_N, S_K\}$ is a martingale, and specifically*

$$\mathbb{E}S_N = \mathbb{E}S_0 = \mathbb{E}S_K.$$

Bounded stopping times, in fact, characterize martingales, as claimed below.

Theorem 8 (Gut Theorem 7.2) *(S_n) is a martingale if and only if $\mathbb{E}S_N =$ constant for every bounded stopping time N.*