

Subgaussian random variables: An expository note

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November 12, 2012

Abstract

In this expository note we explore subgaussian random variables and their basic properties. We also present equivalent formulations of the subgaussian condition, and we discuss briefly the structure of the space of subgaussian random variables. All the results contained here are known; pertinent references are provided wherever possible, though some of the knowledge presented here seems to be ‘folklore’ and the author has abandoned disheartedly the tedious task of tracking down original sources.

1 Introduction

Intuitively, a random variable is called subgaussian when it is subordinate to a Gaussian random variable, in a sense that will be made precise momentarily. As it turns out, subgaussians are a natural kind of random variables for which the properties of Gaussians can be extended ([1]); probably one of the reasons why subgaussians attracted interest in the first place.

To the best of the author’s knowledge, subgaussian random variables were introduced by Kahane in [3], where they played a role to establish a sufficient condition for the almost-sure uniform convergence of certain random series of functions. The name “subgaussian” is the English counterpart of the French “sous-gaussienne” coined by Kahane in [3]. Subsequent works have studied subgaussian random variables and processes either *per se* or in connection with various other subjects. For instance, subgaussian random variables have been studied in connection with random series in [2]; in connection with the geometry of Banach Spaces in [9]; with the spectral properties of random matrices in [7], [12].

2 Subgaussian random variables

A real-valued random variable X is said to be *subgaussian* if it has the property that there is some $b > 0$ such that for every $t \in \mathbb{R}$ one has

$$\mathbb{E} e^{tX} \leq e^{b^2 t^2 / 2}.$$

Thus, the condition for X to be subgaussian says that there is a positive real number b such that the Laplace transform of X is dominated by the Laplace transform of a Gaussian random variable with mean zero and variance b^2 . When this condition is satisfied with a particular value of $b > 0$, we say that X is b -subgaussian, or subgaussian with parameter b .

It is an immediate consequence of this definition that subgaussian random variables are centered, and their variance has a natural upper bound in terms of the subgaussian parameter. We state this “for the records” in the next proposition, whose proof has been borrowed from [13]; although it should be pointed out that this was known much earlier (see e.g. [1]).

Proposition 2.1. *If X is b -subgaussian, then $\mathbb{E}(X) = 0$ and $\text{Var}(X) \leq b^2$.*

PROOF. Using Taylor’s expansion for the exponential function and Lebesgue’s Dominated Convergence Theorem, for any $t \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}(X^n) = \mathbb{E} e^{tX} \leq e^{b^2 t^2 / 2} = \sum_{n=0}^{\infty} \frac{b^{2n} t^{2n}}{2^n n!}.$$

Thus

$$\mathbb{E}(X)t + \mathbb{E}(X^2) \frac{t^2}{2!} \leq \frac{b^2 t^2}{2} + o(t^2) \quad \text{as } t \rightarrow 0.$$

Dividing through by $t > 0$ and letting $t \rightarrow 0$ we get $\mathbb{E}(X) \leq 0$. Dividing through by $t < 0$ and letting $t \rightarrow 0$ we get $\mathbb{E}(X) \geq 0$. Thus $\mathbb{E}(X) = 0$. Now that this is established, we divide through by t^2 and let $t \rightarrow 0$, thus getting $\text{Var}(X) \leq b^2$. ■

Next we look at three natural examples of subgaussian random variables.

Example 2.2. The most natural example of a subgaussian random variable is that of a centered Gaussian. If X has the distribution $\mathcal{N}(0, \sigma^2)$, then an easy computation shows that for any $t \in \mathbb{R}$,

$$\mathbb{E} e^{tX} = e^{\sigma^2 t^2 / 2}.$$

Thus X is subgaussian with parameter σ .

Example 2.3. Let X be a random variable with the Rademacher distribution, meaning that the law of X is $\mathbb{P}_X = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ [here δ_x is the point mass at x]. Then for any $t \in \mathbb{R}$,

$$\mathbb{E} e^{tX} = \frac{1}{2}e^{-t} + \frac{1}{2}e^t = \cosh t \leq e^{t^2/2},$$

so X is 1-subgaussian. Random variables with this distribution are also called symmetric ± 1 random variables, or symmetric Bernoulli random variables.

Example 2.4. Suppose X is uniformly distributed over the interval $[-a, a]$ for some fixed $a > 0$, meaning the law of X is $\mathbb{P}_X = \frac{1}{2a}\mathbf{1}_{[-a, a]}\lambda$, where λ is Lebesgue measure. Then for any real $t \neq 0$,

$$\mathbb{E} e^{tX} = \frac{1}{2a} \int_{-a}^a e^{tx} dx = \frac{1}{2at} [e^{at} - e^{-at}] = \sum_{n=0}^{\infty} \frac{(at)^{2n}}{(2n+1)!}.$$

Using the inequality $(2n+1)! \geq n!2^n$, we see that X is a -subgaussian.

More generally, any centered and bounded random variable is subgaussian, as we demonstrate now (see e.g. [13, Theorem 9.9]).

Theorem 2.5. *If X is a random variable with $\mathbb{E}(X) = 0$ and $|X| \leq 1$ a.s., then*

$$\mathbb{E} e^{tX} \leq \cosh t \quad \forall t \in \mathbb{R} \tag{1}$$

and so X is 1-subgaussian. Moreover, if equality holds in (1) for some $t \neq 0$, then X is a Rademacher variable and hence equality holds for all $t \in \mathbb{R}$.

PROOF. Define f on \mathbb{R} by $f(t) := e^t[\cosh t - \mathbb{E}(e^{tX})]$. Thus

$$f(t) = \frac{1}{2}e^{2t} + \frac{1}{2} - \mathbb{E}(e^{t(1+X)}).$$

For convenience let us set $Y := 1 + X$, so $f(t) = \frac{1}{2}e^{2t} + \frac{1}{2} - \mathbb{E}(e^{tY})$. Apply the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem to conclude $f'(t) = e^{2t} - \mathbb{E}(Ye^{tY})$. Using $\mathbb{E}(Y) = 1$,

$$f'(t) = \mathbb{E}(Y(e^{2t} - e^{tY})).$$

Since $0 \leq Y \leq 2$ a.s., we have

$$t \geq 0 \implies Y(e^{2t} - e^{tY}) \geq 0 \quad a.s.$$

It follows that $f' \geq 0$ and f is increasing on $[0, \infty)$. In particular, for $t \geq 0$ we have $f(t) \geq f(0) = 0$, and so (1) holds for $t \geq 0$. Since $-X$ satisfies the same hypothesis as X , we have just proved that (1) holds for all $t \in \mathbb{R}$.

Now suppose that equality holds in (1) for some $t_0 > 0$. Then $f(t_0) = f(0) = 0$, which implies $f(t) = 0$ for all $t \in [0, t_0]$. Thus $f'(t_0) = 0$, and hence $Y(e^{2t_0} - e^{t_0 Y}) = 0$ a.s. Therefore

$$\mathbb{P}(X = -1) + \mathbb{P}(X = 1) = \mathbb{P}(Y = 0) + \mathbb{P}(Y = 2) = 1.$$

Since $\mathbb{E}(X) = 0$, it follows that $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = 1/2$, and so X is a Rademacher variable. If, on the other hand, equality holds in (1) for some $t_0 < 0$, then applying the same argument to $-t_0 > 0$ and $-X$ we see that $-X$ is a Rademacher variable, and hence so is X . \blacksquare

Corollary 2.6. *If X is a random variable with $\mathbb{E}(X) = 0$ and $|X| \leq b$ a.s. for some $b > 0$, then X is b -subgaussian.*

The set of all subgaussian random variables has a linear structure. The proof that this set is stable under scalar multiples is trivial. For stability under sums the proof we present comes from [1].

Theorem 2.7. *If X is b -subgaussian, then for any $\alpha \in \mathbb{R}$, the random variable αX is $|\alpha|b$ -subgaussian. If X_1, X_2 are random variables such that X_i is b_i -subgaussian, then $X_1 + X_2$ is $(b_1 + b_2)$ -subgaussian.*

PROOF. Suppose X is b -subgaussian. For $\alpha \neq 0$, we have

$$\mathbb{E} e^{t(\alpha X)} \leq e^{b^2 \alpha^2 t^2 / 2} = e^{(|\alpha|b)^2 t^2 / 2}.$$

Now suppose that X_i is b_i -subgaussian, for $i = 1, 2$. For any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, using Hölder inequality,

$$\begin{aligned} \mathbb{E} e^{t(X_1 + X_2)} &\leq \left[\mathbb{E}(e^{tX_1})^p \right]^{1/p} \left[\mathbb{E}(e^{tX_2})^q \right]^{1/q} \\ &\leq \exp\left\{ \frac{t^2}{2} (pb_1^2 + qb_2^2) \right\} = \exp\left\{ \frac{t^2}{2} \left(pb_1^2 + \frac{p}{p-1} b_2^2 \right) \right\}. \end{aligned}$$

Minimizing over $p > 1$ we get

$$\mathbb{E} e^{t(X_1 + X_2)} \leq \exp\left\{ \frac{t^2}{2} (b_1 + b_2)^2 \right\},$$

and the claim follows. \blacksquare

Remark. In the context of Theorem 2.7, if the random variables X_1, X_2 are required to be independent, then the parameter $b_1 + b_2$ can be improved to $\sqrt{b_1^2 + b_2^2}$ (see e.g. [3]).

As it turns out, the set of subgaussian random variables has a much richer structure. For a centered random variable X , the *subgaussian moment* of X , denoted $\sigma(X)$, is defined as follows

$$\sigma(X) := \inf \left\{ b \geq 0 \mid \mathbb{E} e^{tX} \leq e^{b^2 t^2 / 2}, \quad \forall t \in \mathbb{R} \right\}. \quad (2)$$

Clearly X is subgaussian if and only if $\sigma(X) < \infty$. Moreover, the functional $\sigma(\cdot)$ is a norm on the space of subgaussian random variables (upon identification of random variables which are equal almost surely), and this normed space is complete (see e.g. [1]).

Remark. We observe that in case $X \sim \mathcal{N}(0, \sigma^2)$ is a centered Gaussian, then $\sigma(X) = \sigma$. Thus for Gaussian variables the subgaussian moment coincides with the standard deviation.

3 Characterization of subgaussians

According to the definition, a real-valued random variable is subgaussian when its Laplace transform is dominated by the Laplace transform of a centered Gaussian. The following theorem presents equivalent conditions for a random variable to be subgaussian. The calculations used to prove it are well known, the absence of a reference should not be taken as a claim of originality but rather as reflecting the fact that this is folklore knowledge.

Theorem 3.1. *For a centered random variable X , the following statements are equivalent:*

- (1) *Laplace transform condition:* $\exists b > 0, \quad \forall t \in \mathbb{R}, \quad \mathbb{E} e^{tX} \leq e^{b^2 t^2 / 2};$
- (2) *subgaussian tail estimate:* $\exists c > 0, \quad \forall \lambda > 0, \quad \mathbb{P}(|X| \geq \lambda) \leq 2e^{-c\lambda^2};$
- (3) *ψ_2 -condition:* $\exists a > 0, \quad \mathbb{E} e^{aX^2} \leq 2.$

PROOF. (1) \Rightarrow (2) Using Markov's inequality, for any $t > 0$ we have

$$\mathbb{P}(X \geq \lambda) = \mathbb{P}(tX \geq t\lambda) \leq \frac{\mathbb{E} e^{tX}}{e^{t\lambda}} \leq e^{-t\lambda + b^2 t^2 / 2},$$

and minimizing over $t > 0$ we get

$$\mathbb{P}(X \geq \lambda) \leq \inf_{t > 0} e^{-t\lambda + b^2 t^2 / 2} = e^{-\lambda^2 / 2b^2}.$$

Similarly one sees that $\mathbb{P}(X \leq -\lambda) \leq e^{-\lambda^2 / 2b^2}$. Then, using the union bound, we get $\mathbb{P}(|X| \geq \lambda) \leq 2e^{-\lambda^2 / 2b^2}$, and the assertion is proved with $c = 1/(2b^2)$.

(2) \Rightarrow (3) Assuming the subgaussian tail estimate is satisfied with a certain $c > 0$, for any a with $0 < a < c$ we have

$$\begin{aligned}\mathbb{E} e^{aX^2} &\leq 1 + \int_0^\infty 2ate^{at^2} \cdot \mathbb{P}(|X| > t) dt \\ &\leq 1 + \int_0^\infty 2at \cdot 2e^{-(c-a)t^2} dt = 1 + \frac{2a}{c-a}.\end{aligned}$$

Then, by taking a small enough (e.g. $a = c/3$), we get $\mathbb{E} e^{aX^2} \leq 2$.

(3) \Rightarrow (1) Assume that $\mathbb{E} e^{aX^2} \leq 2$ for some $a > 0$. Recalling that X is centered, we have

$$\begin{aligned}\mathbb{E} e^{tX} &= 1 + \int_0^1 (1-y) \mathbb{E}[(tX)^2 e^{y t X}] dy \leq 1 + \frac{t^2}{2} \mathbb{E}[X^2 e^{|tX|}] \\ &\leq 1 + \frac{t^2}{2} e^{t^2/2a} \mathbb{E}[X^2 e^{aX^2/2}] \\ &\leq 1 + \frac{t^2}{2a} e^{t^2/2a} \mathbb{E} e^{aX^2} \\ &\leq \left(1 + \frac{t^2}{a}\right) e^{t^2/2a},\end{aligned}$$

From this it follows that X is subgaussian with parameter $\beta = \sqrt{\frac{3}{a}}$. ■

Remark. If the random variable X has the Gaussian distribution $\mathcal{N}(0, \sigma^2)$, then for each $p > 0$ one has

$$\mathbb{E}|X|^p = \sqrt{\frac{2^p}{\pi}} \sigma^p \Gamma\left(\frac{p+1}{2}\right).$$

In fact, if the random variable X is subgaussian, then its (absolute) moments are bounded above by an expression involving the subgaussian parameter and the gamma function, somewhat similar to the right hand side of the above expression for the moments of a Gaussian (see e.g. [14, p. 93]).

Proposition 3.2. *If X is b -subgaussian, then for any $p > 0$ one has*

$$\mathbb{E}|X|^p \leq p 2^{\frac{p}{2}} b^p \Gamma\left(\frac{p}{2}\right).$$

Consequently, for $p \geq 1$,

$$\|X\|_{L_p} = (\mathbb{E}|X|^p)^{1/p} \leq Cb \sqrt{p}.$$

Conversely, if a centered random variable X satisfies $(\mathbb{E}|X|^p)^{1/p} \leq Cb \sqrt{p}$ for all $p \geq 1$, then X is subgaussian.

PROOF. Assume X is subgaussian, and let $p > 0$. Using the distribution formula and the subgaussian tail estimate,

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1} \mathbb{P}(|X| > t) dt \leq \int_0^\infty pt^{p-1} \cdot 2e^{-t^2/2b^2} dt,$$

using the substitution $u = t^2/2b^2$ the last integral is

$$\begin{aligned} &= p(2b^2)^{\frac{p}{2}} \int_0^\infty u^{\frac{p}{2}-1} e^{-u} du \\ &= p 2^{\frac{p}{2}} b^p \Gamma\left(\frac{p}{2}\right). \end{aligned}$$

In particular, using Stirling's formula one gets $(\mathbb{E}|X|^p)^{1/p} \leq Cb\sqrt{p}$, with $C > 0$ an absolute constant.

Conversely, suppose X satisfies $(\mathbb{E}|X|^p)^{1/p} \leq Cb\sqrt{p}$ for all $p \geq 1$. Then, using the Taylor expansion for the exponential function and Lebesgue's Dominated Convergence Theorem, for any $a > 0$ we have

$$\begin{aligned} \mathbb{E} e^{aX^2} &= \sum_{n=0}^\infty \frac{a^n \mathbb{E}(|X|^{2n})}{n!} = 1 + \sum_{n=1}^\infty \frac{a^n \mathbb{E}(|X|^{2n})}{n!} \\ &\leq 1 + \sum_{n=1}^\infty \frac{a^n (Cb\sqrt{2n})^{2n}}{n!} = \sum_{n=0}^\infty \frac{a^n (Cb\sqrt{2n})^{2n}}{n!} \end{aligned}$$

Taking a small enough one gets $\mathbb{E} e^{aX^2} \leq 2$. This proves that the random variable X satisfies the ψ_2 -condition, so it is subgaussian. \blacksquare

4 The Orlicz space L_{ψ_2}

By ψ_2 we denote the Orlicz function

$$\psi_2(x) = e^{x^2} - 1.$$

The purpose of this section is to construct a special normed space associated to this function, and to give some insight into the ψ_2 -condition. This material is borrowed from [10]. We define

$$\mathcal{L}_{\psi_2} = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} \mid \mathbb{E} \psi_2\left(\frac{|f|}{t}\right) < \infty \text{ for some } t > 0 \right\}.$$

We claim that this is a linear space. For it is clear that the zero function is in \mathcal{L}_{ψ_2} . Given any $f \in \mathcal{L}_{\psi_2}$ and real number $\lambda \neq 0$, let $t > 0$ be such that $\mathbb{E} \psi_2(|f|/t) < \infty$, and set $t' = |\lambda|t$. We have

$$\mathbb{E} \psi_2\left(\frac{|\lambda f|}{t'}\right) = \mathbb{E} \psi_2\left(\frac{|f|}{t}\right) < \infty,$$

which proves that $\lambda f \in \mathcal{L}_{\psi_2}$. Finally, if $f, g \in \mathcal{L}_{\psi_2}$, choose $t, s > 0$ such that $\mathbb{E} \psi_2(|f|/t) < \infty$ and $\mathbb{E} \psi_2(|g|/s) < \infty$. Since the function ψ_2 is increasing and convex, we have

$$\begin{aligned} \psi_2\left(\frac{|f+g|}{t+s}\right) &\leq \psi_2\left(\frac{|f|+|g|}{t+s}\right) \\ &\leq \frac{t}{t+s} \psi_2\left(\frac{|f|}{t}\right) + \frac{s}{t+s} \psi_2\left(\frac{|g|}{s}\right). \end{aligned}$$

Then, taking expectations,

$$\mathbb{E} \psi_2\left(\frac{|f+g|}{t+s}\right) \leq \frac{t}{t+s} \mathbb{E} \psi_2\left(\frac{|f|}{t}\right) + \frac{s}{t+s} \mathbb{E} \psi_2\left(\frac{|g|}{s}\right). \quad (3)$$

Since the right hand side is finite, we see that $f+g \in \mathcal{L}_{\psi_2}$.

We define a functional $\|\cdot\|_{\psi_2} : \mathcal{L}_{\psi_2} \rightarrow \mathbb{R}$ by setting

$$\|f\|_{\psi_2} = \inf \left\{ t > 0 \mid \mathbb{E} \psi_2\left(\frac{|f|}{t}\right) \leq 1 \right\}. \quad (4)$$

Given $f \in \mathcal{L}_{\psi_2}$, choose $t > 0$ such that $\mathbb{E} \psi_2(|f|/t) < \infty$. Since ψ_2 is increasing for positive values, it follows that $\mathbb{E} \psi_2(|f|/s) < \infty$ for all $s \geq t$. Then, using Lebesgue's Dominated Convergence Theorem,

$$\lim_{s \rightarrow \infty} \mathbb{E} \psi_2\left(\frac{|f|}{s}\right) = 0.$$

This implies that there is some $t_0 > 0$ such that $\mathbb{E} \psi_2(|f|/t_0) \leq 1$. Thus we have proved that $\|f\|_{\psi_2} < \infty$ for $f \in \mathcal{L}_{\psi_2}$, showing that $\|\cdot\|_{\psi_2}$ is well defined. It is clear that $\|\cdot\|_{\psi_2} \geq 0$.

If $f = 0$ a.e., then clearly $\|f\|_{\psi_2} = 0$. Conversely, let $f \in \mathcal{L}_{\psi_2}$ be such that $\|f\|_{\psi_2} = 0$. It follows that $\mathbb{E} \psi_2(n|f|) \leq 1$ for all $n \geq 1$. Assuming $f \neq 0$ a.e., that is, $\mathbb{P}(|f| > 0) > 0$, we may find some positive real number δ such that the event $A := \{\omega \in \Omega \mid |f(\omega)| \geq \delta\}$ has $\mathbb{P}(A) > 0$. Then we have

$$\psi_2(n\delta) \mathbb{P}(A) = \int_A \psi_2(n\delta) d\mathbb{P} \leq \int_A \psi_2(n|f|) d\mathbb{P} \leq \mathbb{E} \psi_2(n|f|) \leq 1,$$

and letting $n \rightarrow \infty$ we get a contradiction. Hence $f = 0$ a.e.

It is clear that if $f \in \mathcal{L}_{\psi_2}$ and $\lambda \in \mathbb{R}$, then $\|\lambda f\|_{\psi_2} = |\lambda| \cdot \|f\|_{\psi_2}$. This is obvious for $\lambda = 0$, and for $\lambda \neq 0$ it follows from properties of the infimum. Thus the functional $\|\cdot\|_{\psi_2}$ is (positively) homogeneous.

To see that $\|\cdot\|_{\psi_2}$ satisfies the triangle inequality, let $f, g \in \mathcal{L}_{\psi_2}$, and choose $t, s > 0$ such that $\mathbb{E} \psi_2(|f|/t) \leq 1$ and $\mathbb{E} \psi_2(|g|/s) \leq 1$. Using inequality (3), we obtain

$$\mathbb{E} \psi_2\left(\frac{|f+g|}{t+s}\right) \leq \frac{t}{t+s} \mathbb{E} \psi_2\left(\frac{|f|}{t}\right) + \frac{s}{t+s} \mathbb{E} \psi_2\left(\frac{|g|}{s}\right) \leq 1.$$

Thus $\|f+g\|_{\psi_2} \leq t+s$, and taking infimum one at a time over t and s we get $\|f+g\|_{\psi_2} \leq \|f\|_{\psi_2} + \|g\|_{\psi_2}$.

We have established that $\|\cdot\|_{\psi_2}$ is a seminorm on \mathcal{L}_{ψ_2} . Upon identifying functions in \mathcal{L}_{ψ_2} which are equal almost everywhere we obtain a normed space denoted $(L_{\psi_2}, \|\cdot\|_{\psi_2})$ and called the Orlicz space associated to the Orlicz function ψ_2 . Accordingly, the functional $\|\cdot\|_{\psi_2}$ is called an Orlicz norm. As is done with the Lebesgue spaces L_p , we regard the elements of L_{ψ_2} as functions, thus avoiding the awkward treatment of ‘classes of functions’ and ‘representatives’ and so on. However, we should keep in mind that equality of elements in L_{ψ_2} means equality almost everywhere.

Remark. This construction works as well for more general functions than ψ_2 . If a function $\psi : [0, \infty) \rightarrow \mathbb{R}$ is continuous and convex with $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$ (necessarily ψ is increasing and $\lim_{t \rightarrow \infty} \psi(t) = \infty$), then one may construct an Orlicz norm $\|\cdot\|_{\psi}$ associated to this function,

$$\|f\|_{\psi} = \inf \left\{ t > 0 \mid \mathbb{E} \psi\left(\frac{|f|}{t}\right) \leq 1 \right\},$$

and an Orlicz space L_{ψ} , following pretty much the construction we presented above for $\psi_2(x) = e^{x^2} - 1$. The reader is referred to [10] to see the details. More about Orlicz norms and spaces can be seen in [5] or [6].

The next proposition, last of this note, helps elucidate the link between subgaussian random variables, the ψ_2 -condition, and the Orlicz space L_{ψ_2} .

Proposition 4.1.

$$\|f\|_{\psi_2} \leq 1 \quad \text{if and only if} \quad \mathbb{E} \psi_2(|f|) \leq 1 \quad \text{if and only if} \quad \mathbb{E} e^{f^2} \leq 2.$$

PROOF. The first equivalence is clear if $\|f\|_{\psi_2} = 0$; and in case $\|f\|_{\psi_2} > 0$, setting $a := \|f\|_{\psi_2}$ we first note that

$$\mathbb{E} \psi_2\left(\frac{|f|}{a}\right) \leq 1.$$

If $a \leq 1$, then $\mathbb{E} \psi_2(|f|) \leq 1$ by the monotonicity of the function ψ_2 . Conversely, if $\mathbb{E} \psi_2(|f|) \leq 1$, then $1 \in \{t > 0 \mid \mathbb{E} \psi_2(|f|/t)\}$, so upon taking the infimum of this set we get $a \leq 1$.

The second equivalence is obvious. ■

Comparing Proposition 4.1 and the ψ_2 -condition from Theorem 3.1, it is now evident that a random variable is subgaussian precisely when it is an element of the Orlicz space L_{ψ_2} .

References

- [1] V.V. Buldygin and Yu.V. Kozachenko, Subgaussian random variables. *Ukrainian Math. J.* **32** (1980), 483 - 489.
- [2] Y.S. Chow, Some Convergence Theorems for Independent Random Variables. *Ann. Math. Statist.* **37** (1966), 1482 - 1493.
- [3] J.P. Kahane, Propriétés locales des fonctions à séries de Fourier aléatoires. *Stud. Math.* **19** (1960), 1 - 25.
- [4] J.P. Kahane, *Some random series of functions*. 2nd ed. Cambridge University Press, London, 1985.
- [5] M.A. Krasnosel'skii and J.B. Rutickii, *Convex Functions and Orlicz Spaces*. Noordhoff, Groningen, 1961.
- [6] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II*. Reprint of the 1977, 1979 ed. Springer, Berlin Heidelberg New York, 1996.
- [7] A.E. Litvak, A. Pajor, M. Rudelson and N. Tomczak-Jaegermann, Smallest singular value of random matrices and geometry of random polytopes. *Adv. Math.* **195** (2005), 491–523.
- [8] M.B. Marcus and G. Pisier, *Random Fourier series with applications to harmonic analysis*. Princeton University Press, Princeton, 1981.
- [9] G. Pisier, Probabilistic Methods in the Geometry of Banach Spaces. In: G. Letta, M. Pratelli, *Probability and Analysis*, Lecture Notes in Mathematics 1206, pp. 167 - 241. Springer-Verlag, Berlin, 1986.
- [10] M.M. Rao, *Theory of Orlicz Spaces*. M. Dekker, New York, 1991.
- [11] M. Rudelson, *Probabilistic and Combinatorial Methods in Analysis*. Unpublished lecture notes, CBMS Regional Meeting, Kent, 2006.
- [12] M. Rudelson and R. Vershynin, The smallest singular value of a random rectangular matrix. *Comm. Pure Appl. Math.* **62** (2009), 1707–1739.

- [13] K.R. Stromberg, *Probability for Analysts*. Chapman & Hall/CRC, New York, 1994.
- [14] D.W. Stroock, *Probability Theory, An analytic view*. 2nd. ed. Cambridge University Press, Cambridge, 2011.

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