

Slide 1

36-310 Spring 2003: Conditioning

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January 23, 2003

- Conditioning and tree diagrams for counting
- Conditioning and tree diagrams for probability

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Conditioning

Conditioning is one of the most important ideas in probability and statistics. It is important for two reasons:

- *Breaking up calculations.* Many counting and probability calculations that look quite difficult can be greatly simplified by conditioning.
- *Prediction and causation.*
 - Statistics makes predictions through conditional probabilities: “if B is true, then A is more likely to occur.”
 - Causal arguments (“B causes A”) can be made when the conditional probability of A depends on B, and B can be *manipulated*.

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Conditional counting

Many probability experiments have outcomes that are obtained by a sequence of actions:

First roll a die	6 ways
Then draw a card	52 ways
Then flip a coin	<u>2 ways</u>
Total	$6 \times 52 \times 2 = 624$ ways

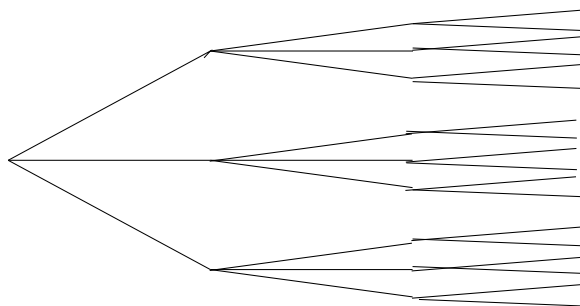
It is not hard to see that if there are n_1 ways to do the first action, and for each of these there are n_2 ways to do the second, and for each of these there are n_3 ways to do the third, **etc.** then the total number of ways to obtain an outcome is

$$n_1 \times n_2 \times n_3 \text{ (and so forth, for more than three actions)}$$

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Tree diagrams

- Can help you see the general principle
- Can help you organize counting for particular problems



First action (3)

Second action (3)

Third Action (2)

Total = $3 \times 3 \times 2 = 18$

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Permutations

The principle can be applied (without explicitly drawing the tree) to the problem of counting *permutations*.

- Any ordered sequence of k objects taken from a set of n distinct objects is a *permutation*
- The *number of permutations* of n objects taken k at a time is denoted P_{kn} .

It is easy to see that

$$P_{kn} = n(n-1)(n-2) \cdots (n-k+1)$$

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Factorials

- The special case P_{nn} is important:

$$P_{nn} = n(n-1)(n-2) \cdots (2)(1) = n!$$

which we pronounce "*n factorial*". To keep formulas simple we also say $0! = 1$.

- With a little algebra we can see that

$$\begin{aligned} P_{kn} &= n(n-1)(n-2) \cdots (n-k+1) \\ &= \frac{n(n-1)(n-2) \cdots (1)}{(n-k)(n-k-1) \cdots (2)(1)} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

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Combinations

Another important basic building block for counting. . .

- Any unordered set of k objects taken from n distinct objects is a *combination*
- The number of combinations of n objects taken k at a time is denoted

$$C_{kn} = \binom{n}{k}$$

From past courses you may remember that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = P_{kn}/k!$$

or, put another way, $P_{kn} = C_{kn} \times P_{kk}$.

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Why does $P_{kn} = C_{kn} \times P_{kk}$?

Task: Construct a permutation of k things from n . We know the number of ways is P_{kn} . But we can count it “conditionally as well:

- | | |
|---|--------------------------|
| 1. Take a subset of k things from n | C_{kn} ways |
| 2. Take a permutation of k things from k | P_{kn} ways |
| TOTAL | $C_{kn} \times P_{kk}$. |

This is a way of *proving* that in fact

$$C_{kn} = \binom{n}{k} = P_{kn}/P_{kk} = \frac{n!}{(n-k)!k!}$$

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Example A real estate agent is showing homes to a prospective buyer. There are 10 homes in the buyer's price range but the buyer only has time to visit 3 of them.

- In how many ways can 3 homes be chosen to visit, if the order of the visits is being considered?
- In how many ways can 3 homes be chosen to visit, if the order is not being considered?
- If four of the homes are new and six have been previously occupied, and the three homes to be visited are randomly chosen, what is the probability that all three are new? [*you should get the same answer whether order is considered or not*]

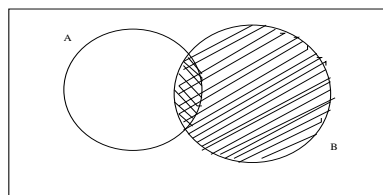
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Conditional Probability

- The conditional probability of A given B is a way of revising the probability of A once you have the new information that B occurred. It is the fraction of probability in B that is accounted for by A.

Definition If $P(B) > 0$, the *conditional probability of A given B* is defined to be

$$P[A|B] = \frac{P[A \cap B]}{P(B)}$$



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Joint Probability Table:

In a certain class...

		HSS		Total
		Yes	No	
Fresh	Yes	0.55	0.10	0.65
	No	0.15	0.20	0.35
Total		0.70	0.30	1.00

Now it is easy to see for example that

- $P[\text{Fresh}|\text{HSS}] = \frac{0.55}{0.70} = \frac{P[\text{Fresh} \cap \text{HSS}]}{P[\text{HSS}]} = 0.79$ is a column percent; and
- $P[\text{HSS}|\text{Fresh}] = \frac{P[\text{Fresh} \cap \text{HSS}]}{P[\text{Fresh}]} = \frac{0.55}{0.65} = 0.85$ is a row percent.

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Multiplication Rule for Conditional Probability

- Since

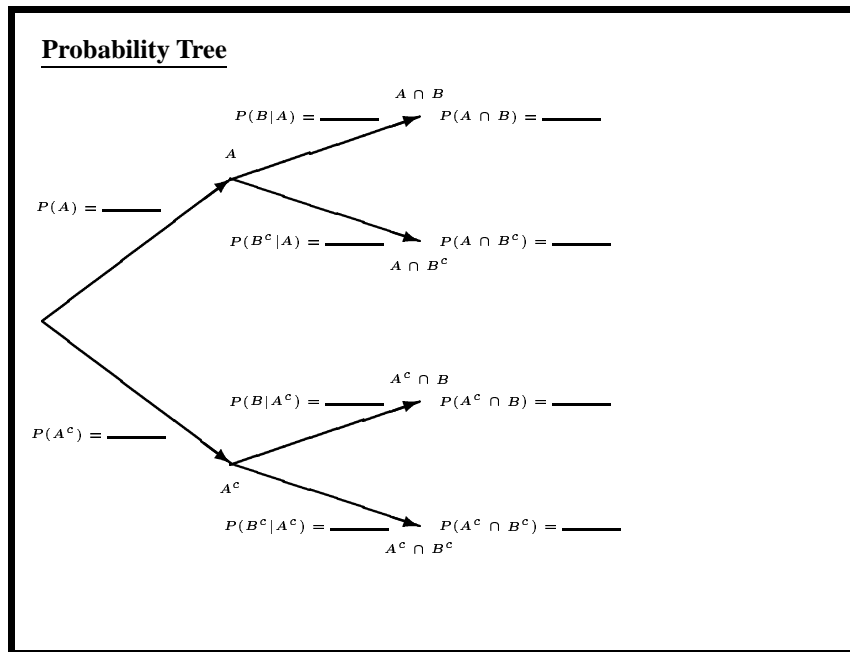
$$P[A|B] = \frac{P[A \cap B]}{P(B)}$$

it follows immediately that

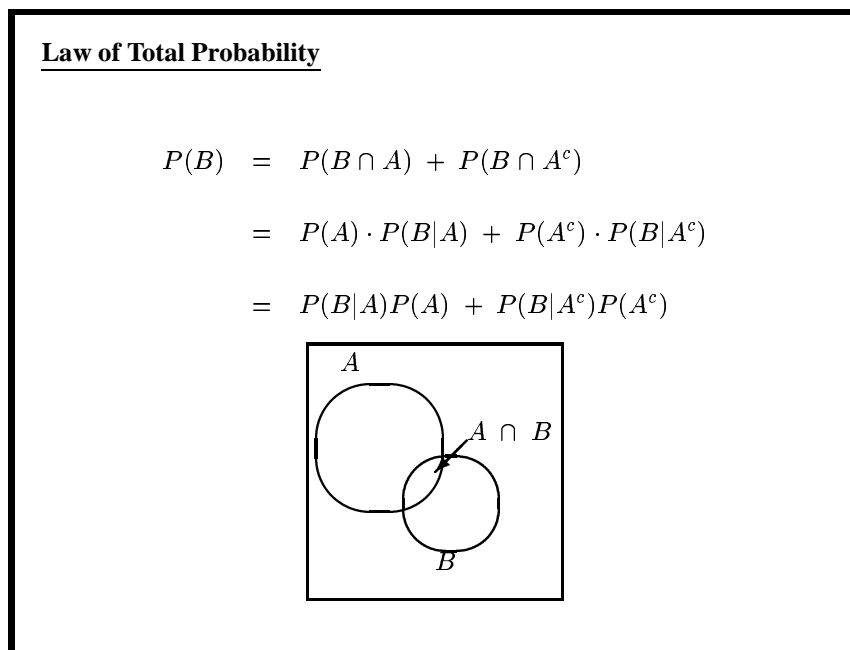
$$P[A \cap B] = P[A|B]P[B] = P[B|A]P[A]$$

- Many combinations (intersections) of events in probability can be viewed as a *sequence of actions*.
 - A tree diagram can again help organize thinking about the sequence of actions.
 - Conditional probabilities can be multiplied “along branches of the tree” to compute probabilities of intersections.

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Bayes' Rule

We know that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

and we can apply the *law of total probability* to B to get

Bayes' Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

It is useful for “turning conditional probabilities around”.

- You are given information about $P(B|A)$ and $P(B|A^c)$
- You want to know $P(A|B)$.
- Use Bayes' Rule!