36-720: Graphical Models

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- Collapsibility and Simpson's Paradox
- Decomposability
- Examples

One nice feature of graphical models is that they lead immediately to two imporant ideas in simplifying log-linear models. They are

- Decomposable models
- Collapsible models

Decomposable models are graphical models for which closed-form MLE's exist.

Collapsibility is closely related to (the non-occurrence of) Simpson's paradox like phenomena.

Collapsibility and Simpson's Paradox

Consider a hierarchical, graphical log-linear model $\mathcal{M}$ and partition the factors in $\mathcal{M}$ into three dijoint groups, A, B, and C. *Examples:*

- $\mathcal{M} = [12][23]$, $A = \{1\}$, $B = \{3\}$, $C = \{2\}$
- $\mathcal{M} = [123][145]$, $A = \{2, 3\}$, $B = \{4, 5\}$, $C = \{1\}$
- $\mathcal{M} = [126][234][46][56]$, $A = \{1, 2, 6\}$, $B = \{3, 5\}$, $C = \{4\}$

Often we are interested in the relationship between A and C

- Conditioning on B, vs.
- Summing (collapsing) over B.

Simpson's paradox happens when

$$P(A|C, B) \neq P(A|C)$$

[e.g. if A and C are single factors, $OR(A, C) \neq OR(A, C|B)$.]

We will say “the conditional relationship $A|C$ is collapsible over B” if

$$P(A|C, B) = P(A|C)$$

Clearly this can happen if and only if

$$B \perp\!\!\!\perp A \mid C$$

Since $\mathcal{M}$ is a graphical model, the *global Markov property* tells us this can happen iff

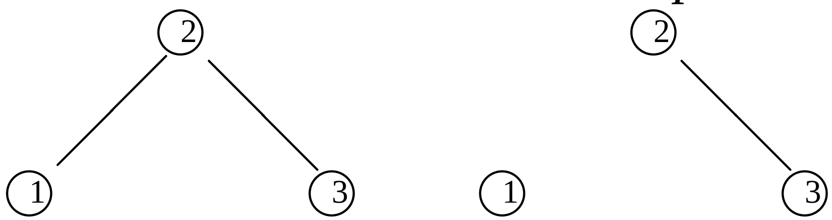
C separates $\mathcal{M}$ into disjoint subgraphs containing A and B.

and when we look at the graph we see this happens iff

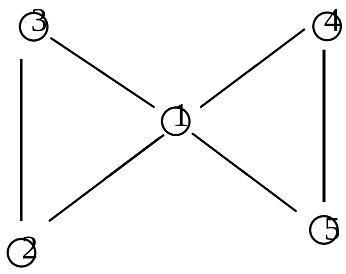
Every path from a variable in A to a variable in B goes through at least one variable in C.

Thus we have a simple criterion for the collapsibility of relationships between groups of variables, over a “stratification” variable. *Examples:*

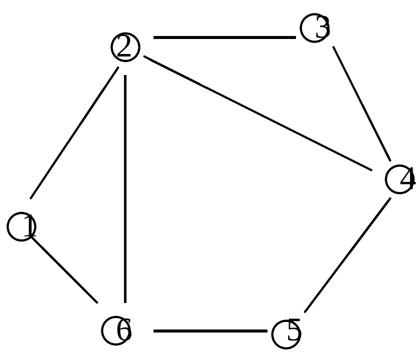
$\mathcal{M} = [12][23]$, $A = \{1\}$, $B = \{3\}$, $C = \{2\}$



$\mathcal{M} = [123][145]$, $A = \{2, 3\}$, $B = \{4, 5\}$, $C = \{1\}$



$\mathcal{M} = [126][234][45][56]$, $A = \{1, 2, 6\}$, $B = \{3, 5\}$, $C = \{4\}$



An Equivalent Form of Collapsibility

Sometimes we are not so much interested in the relationship between variables in A and those in C, but rather in *all* the relationships in A, afer collapsing over B.

Let A be a subset of variables in $\mathcal{M}$ and let $B = \mathcal{M} \setminus A$.

We say $\mathcal{M}$ *is collapsible* **onto** A iff the models

- $P(A)$
- $P(A|B)$

have the same graph [namely, the graph of $\mathcal{M}$ with the variables in B deleted]. I.e., $P(A)$ has exactly the same conditional independence relationships as $P(A|B)$.

The two forms of collapsibility are equivalent:

- Let $A = A' \cup C'$ [disjoint union]. Clearly, if $\mathcal{M}$ is collapsible onto A, then $P(A'|C') = P(A'|C', B)$ since these are determined by $P(A)$ and $P(A|B)$.
- Conversely if $A'|C'$ is collapsible over B, then the table can be collapsed onto $A = A' \cup B'$. This follows from Theorem 2.3 and Corollary 2.5 of Asmussen & Edwards (1983, *Bmka*), which also establishes the criterion on the next slide.

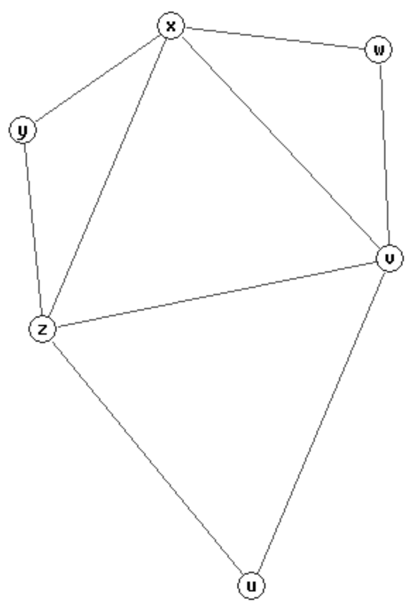
A criterion for collapsibility of $\mathcal{M}$ onto A

Let

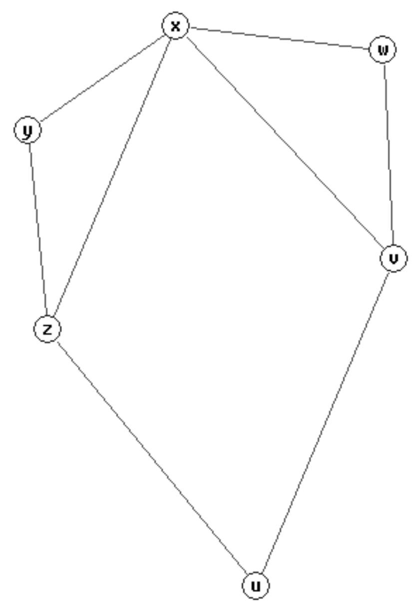
- $A \subseteq \mathcal{M}$, $B = \mathcal{M} \setminus A$
- $B_1, \ldots, B_k$ be the connected components of B
- $E_1, \ldots, E_k$ be their boundaries

Then $\mathcal{M}$ is collapsible onto A iff each E_j is complete.

Example



Model $\mathcal{M}$, $A = \{X, Y, Z, V\}$



$\mathcal{M}$, $A = \{X, Y, Z, V\}$

Decomposability

The idea of decomposability is to reduce the model formulae for $f \in \mathcal{M}$ to a sequence of saturated models. This allows:

- Closed-form MLE's, for log-linear models (and other cases);
- Simple interpretations;
- Suggestions of a causal or time-order sequence for the variables.

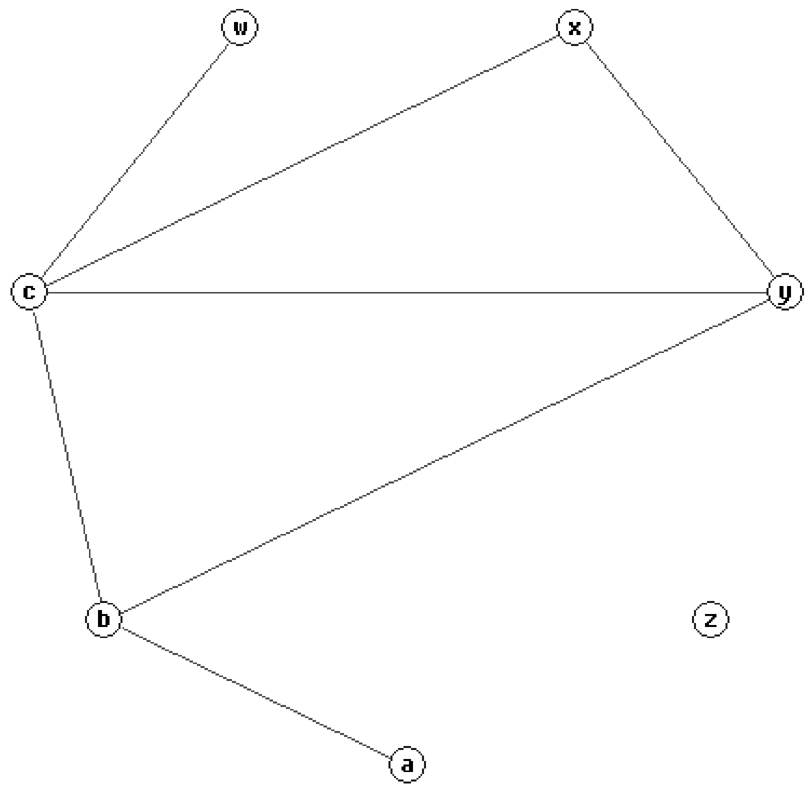
Let $\mathcal{M}$ be a graphical log-linear model.

- Let $c_1, \ldots, c_k$ be all the cliques in $\mathcal{M}$
- Let $b_2 = c_2 \cap c_1$, $b_3 = c_3 \cap (c_2 \cup c_1)$, $\ldots$, $b_k = c_k \cap (\cup_{t=1}^{k-1} c_t)$ be the "running intersections sets"

The clique ordering $c_1, \ldots, c_k$ is an *SD-clique ordering* if, for every $s = 2, 3, \ldots, k$, there is a $j < s$ such that $b_s \subseteq c_j$.

The graphical model $\mathcal{M}$ is *decomposable* if there is an SD-clique ordering.

Example



The cliques are $c_1 = \{A, B\}$, $c_2 = \{B, C, Y\}$, $c_3 = \{C, W\}$, $c_4 = \{C, X, Y\}$, $c_5 = \{Z\}$.

The running intersection sets are $d_2 = \{B\}$, $d_3 = \{C\}$, $d_4 = \{C, Y\}$, $d_5 = \emptyset$.

Some properties of decomposible graphs

- Decomposible graphs are collapsible.
  - In fact, $\mathcal{M}$ is collapsible onto each $u_s = \cup_{t=1}^{s} c_t$, $s < k$;
  - Clearly, each u_s is also collapsible, onto each u_r, $r < s$;
- Using the relation $f = f_{b|a} f_a$ recursively we have that

  $$f = f(c_1) f(c_2|u_1) f(c_3|u_2) \cdots f(c_k|u_{k-1})$$

  for all $f \in \mathcal{M}$.
- Thus to find the MLE $\hat{f}$ for a decomposible log-linear graphical model we should fit the *saturated model* $f(c_s|u_{s-1})$ to each clique, and then multiply the fitted models together.
- MLE's are not much harder to find for general decomposible models.
- It can be shown that log-linear graphical models are decomposible iff they are *triangulated*. (A stronger condition is needed if both discrete and continuous nodes are present). See next slides.

- It turns out that decomposable graphical models are exactly those models that are "triangulated":
  - A *cycle* is a sequence of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ in $\mathcal{M}$ such that $v_1, \ldots, v_{k-1}$ are distinct vertices, and $v_1 = v_k$. A *chord* is an edge (v_i, v_j) in $\mathcal{M}$ between non-adjacent vertices ($|i - j| > 1$) in a cycle.
  - A *triangulated graph* is one whose *chordless cycles* contain no more than three vertices.

- For a triangulated graphical model it is easy to read off the MLE's:
  - The cell counts will have the form

$$m_{ijklpqr(etc.)} = \frac{\prod(\textit{minimal sufficient margins})}{\prod(\textit{separator margins})}$$

    where

    * The *"minimal sufficient margins"* are the margins with fixed indices corresponding to the terms in the generator (cliques in the graph!); and
    * the *"separator margins"* are the margins with fixed indices corresponding to variables common to terms in the generator (these terms are *minimal* sets of vertices separating the graph into disconnected parts).

Examples

- All log-linear models for three-way tables are decomposable, except for the model of no three-way interaction (which is not even graphical).
- [12][26][235][345]
  - Graph is triangulated
  - The separator sets are [2] (with multiplicity 2) and [35] (multiplicity 1)

$$m_{ijklrq} = \frac{m_{ij++++}m_{+j+++r}m_{+jk+q+}m_{++klq+}}{(m_{+j++++})^2 m_{++k+q+}}$$

- [12][26][235][345][245] not decomposable—not even graphical!
- [126][234][45][56] is graphical but not triangulated.
- [126][234][456][246] is graphical and triangulated. What are the MLE's $\hat{m}_{ijklrq}$?