36-720: Discrete Multivariate Analysis
HW02, Due Wednesday, September 19, 2005

Announcements:

- We are finishing up with the first part of the course, Chapters 2, 3 and parts of Chapters 9 and 10, in Christensen. Next week we will turn to graphical models (parts of Chapter 5, especially sections 5.1–5.3) and model selection/criticism (parts of Chapter 6); Chapters 2 and 4 of Edwards are also relevant.
- Problems #4 – #7 below were written somewhat quickly. Please solve the "intended" problem if there are flaws!
- Please remember that, although you are free to talk with one another about HW's, the work you turn in should be your own.
- Best to use a word processor that can handle mathematics (like LaTeX) and can include graphics from R and other programs. Next best is neat handwriting with neatly cut-and-pasted tables and graphs.

Problems:

1. p. 113, Ex. 3.8.2
2. p. 114, Ex. 3.8.7 Iterative Proportional Fitting is scalable to large tables, especially when the log-linear model is not too complicated, so even though it is an old method, it is worth knowing something about.
3. pp. 113–114, Ex. 3.8.4
4. pp. 114–115, Ex. 3.8.9

   In addition, please do:

   (d) Use the model of no three-way interaction to test whether there is a common odds ratio for sex and admission in all six departments.

   (e) Whether or not you accepted the hypothesis of a common odds ratio, use the no three-way interaction model to estimate the common odds ratio and give a confidence interval for it. Compare with a confidence interval for the odds ratio in the marginal table for sex and admission (summed across departments).

   The Mantel-Haenszel test is still quite popular in many settings. For example it is often used to test for the effect of treatement on outcome in multicenter clinical trials where the $2 \times 2 \times K$ table is (treatment)$\times$(outcome)$\times$(center); and it is used to assess test-question bias by major testing organizations where the $2 \times 2 \times K$ table is (right/wrong)$\times$(majority/minority group)$\times$(total score on other questions).

[Continued on back]

5. Let $n = (n_{ij} : i = 1 \ldots I, j = 1 \ldots J)$ be a two-way table of counts that follows the Poisson sampling model. Use the delta method to show that the asymptotic standard error of the estimated log-odds ratio

$$\log \widehat{OR} = \log \frac{n_{ij} n_{i'j'}}{n_{i'j} n_{ij'}} \qquad (*)$$

is

$$\sqrt{\frac{1}{n_{ij}} + \frac{1}{n_{i'j'}} + \frac{1}{n_{i'j}} + \frac{1}{n_{ij'}}}$$

6. Let $n = (n_1, \ldots, n_C)$ be a table with total $N = n_+$, following the multinomial sampling model: $n \sim Multinom(N, p)$ where $p = (p_1, \ldots, p_C)$ are the cell probabilities.

   (a) For $N = 1$, compute the variance-covariance matrix V for n.

   (b) Show that, as $N \to \infty$,

   $$(n - m)/\sqrt{N} \sim N(0, V),$$

   where $m = E[n]$, by showing that a standard asymptotic theorem applies, and gives this result. (A more general theory is sketched in Chapter 12 of Christensen.)

   (c) Now assume that the n_c's can be re-indexed as n_{ij} in a two-way table. Use the delta method to obtain the asymtotic standard error of $\log \widehat{OR}$ in $(*)$ again, this time under multinomial sampling.

7. Suppose $n = (n_1, \ldots, n_C)$ is sampled from an exponential family model with parameter vector θ

$$L(n \mid \theta) = G(n) e^{B(\theta) + K(n)^T \gamma(\theta)}$$

where $K(n) = (K_1(n), \ldots, K_d(n))$ are the *sufficient statistics*, and $\gamma(\theta) = (\gamma_1(\theta), \ldots, \gamma_d(\theta))$ are the *natural parameters*. Let $\ell(\theta) = \log L(n \mid \theta)$.

   (a) Compute the gradient $\partial \ell(\theta)/\partial\theta$ and show that

   $$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{\partial B(\theta)}{\partial \theta} + K(n)^T \frac{\partial \gamma(\theta)}{\partial \theta}$$

   where $\frac{\partial B(\theta)}{\partial \theta}$ is the gradient of $B(\theta)$ and $\frac{\partial \gamma(\theta)}{\partial \theta}$ is an appropriate matrix of partial derivatives.

   (b) Use part (a) together with the fact that $E[\partial \ell(\theta)/\partial\theta \mid \theta] = 0$ to show that

   $$\frac{\partial \ell(\theta)}{\partial \theta} = [K(n) - \mu(\theta)]^T \frac{\partial \gamma(\theta)}{\partial \theta}$$

   where $\mu(\theta) = E[K(n)]$ is the vector of expected sufficient statistics.

   (c) Now suppose $d = C$, $K(n) = (n_1, \ldots, n_C)$, and $\gamma(\theta) = A\theta$. Show that, for the MLE $\hat{\theta}$,

   $$\mu(\hat{\theta})^T A = n^T A$$

   (d) What does this have to do with equating expected and observed margins for the MLE of a log-linear model?