Latent and manifest monotonicity in item response models *

Brian W. Junker
Department of Statistics
Carnegie Mellon University
Pittsburgh PA, USA
http://www.stat.cmu.edu/~brian
brian@stat.cmu.edu

Klaas Sijtsma
Vakgroep Methoden en Technieken van Onderzoek
Faculteit der Sociale Wetenschappen
Tilburg University
Tilburg, The Netherlands
k.sijtsma@kub.nl

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Motivation and Background

Monotone Unidimensional IRT model

- $X_1, X_2, \ldots$ binary items

- $P_j(\theta) = P[X_j = 1|\theta]$ *nondecreasing* in θ

- θ unidimensional with density or pmf $f(\theta)$

$$
\begin{aligned}
& P[X_1 = x_1, \ \ldots, \ X_J = x_J] \\
& \qquad = \int \prod_{j=1}^{J} P_j(\theta)^{x_j} (1 - P_j(\theta))^{1-x_j} f(\theta) d\theta \quad (1)
\end{aligned}
$$

- *Main idea:* Use $P[X_j = 1|\hat{\theta}]$ to check monotonicity of $P_j(\theta)$

Why Check Monotonicity?

- Precursor to parametric model fitting

- Nonparametric scaling / item analysis

- Nonparametric inference: If the model holds then may use

$$P[\theta > t|S = s] \textit{ is nondecreasing in } s \textit{ for each } t. \qquad (SOL)$$

to make inferences (e.g. *mastery*) on θ nonparametrically, where

$$S = \sum_{j=1}^{J} X_j$$

is the *total score*.

(Grayson, 1988; Huynh, 1994; Hemker, Sijtsma, Molenaar, and Junker, 1996, 1997)

How to Check Monotonicity?

Checking monotonicity of the *item-total regression*

$$P[X_j = 1|S = s]$$

has been suggested or explored by

- Sijtsma (1988) — Dissertation on Mokken's IRT model.
- Ramsay (1991) — EDA for IRT using kernel-smoothed estimate of $P[X_j = 1|S = s]$.
- Croon (1991) — Constrains ordered latent class fits to make $P[X_j = 1|S = s]$ monotone.
- Scheiblechner (1995) — Manifest condition for ISOP model.

Intuitively, one expects that conditioning on $S = s$ will make $P[X_j = 1|S = s]$ "too monotone".

Reason:

- We usually correct $Corr(X_j, S)$ by replacing S with the *rest-score*

$$S_{(-j)} = S - X_j$$

since $Corr(X_j, S)$ is expected to be too large.

Does this intuition really work?

Some positive results

Theorem (Junker, 1993, 1996; others?)

- Under the model (1), $P[X_j = 1|S_{(-j)} = r]$ is nondecreasing in r.
- Under the Rasch model, $P[X_j = 1|S = r]$ is also nondecr. in r.

Proof Sketch:

Let R stand for either S or $S_{(-j)}$. We want to show

$$\left.\begin{aligned} P[X_j = 1|R = r-1] &= \int P[X_j = 1|R = r-1, \theta]\, f(\theta|R = r-1) d\theta \\ &\stackrel{?}{\leq} \int P[X_j = 1|R = r, \theta]\, f(\theta|R = r-1) d\theta \\ &\stackrel{?}{\leq} \int P[X_j = 1|R = r, \theta]\, f(\theta|R = r) d\theta \\ &= P[X_j = 1|R = r], \end{aligned}\right\} \quad (2)$$

- *First* "$\stackrel{?}{\leq}$": Sufficient to show
  - $P[X_j = 1|R = r, \theta]$ nondecr. in r for each θ
- *Second* "$\stackrel{?}{\leq}$": Sufficient to show
  - θ stochastically ordered by R;
  - $P[X_j = 1|R = r, \theta]$ nondecr. in θ for each r $\square$

Proof for $R = S_{(-j)}$, monotone IRT model (1):

$$\begin{aligned} P[X_j = 1|S_{(-j)} = r-1] &= \int P[X_j = 1|S_{(-j)} = r-1, \theta]\, f(\theta|S_{(-j)} = r-1)d\theta \\ &\overset{?}{\leq} \int P[X_j = 1|S_{(-j)} = r, \theta]\, f(\theta|S_{(-j)} = r-1)d\theta \\ &\overset{?}{\leq} \int P[X_j = 1|S_{(-j)} = r, \theta]\, f(\theta|S_{(-j)} = r)d\theta \\ &= P[X_j = 1|S_{(-j)} = r], \end{aligned}$$

- *First* "$\overset{?}{\leq}$":
  - By LI, $P[X_j = 1|S_{(-j)} = r, \theta] = P_j(\theta)$ nondecreasing in r for each θ
- *Second* "$\overset{?}{\leq}$":
  - Grayson (1988); Huynh (1994): θ stochastically ordered by $S_{(-j)}$;
  - By LI, $P[X_j = 1|S_{(-j)} = r, \theta] = P_j(\theta)$, nondecreasing in θ for each r $\square$

Proof for $R = S$, Rasch model:

$$
\begin{aligned}
P[X_j = 1|S = r-1] &= \int P[X_j = 1|S = r-1, \theta]\, f(\theta|S = r-1) d\theta \\
&\stackrel{?}{\leq} \int P[X_j = 1|S = r, \theta]\, f(\theta|S = r-1) d\theta \\
&\stackrel{?}{\leq} \int P[X_j = 1|S = r, \theta]\, f(\theta|S = r) d\theta \\
&= P[X_j = 1|S = r],
\end{aligned}
$$

- <u>*First*</u> "$\stackrel{?}{\leq}$":
  - *In general* $P[X_j = 1|S = r, \theta]$ nondecreasing in r for each θ (Scheiblechner 1995, Theorem 4)
- <u>*Second*</u> "$\stackrel{?}{\leq}$":
  - Grayson (1988); Huynh (1994): θ stochastically ordered by S;
  - By sufficiency (Rasch), $P[X_j = 1|S = r, \theta] =$ $P[X_j = 1|S = r]$, nondecreasing in θ for each r $\square$

$\underline{P[X_j = 1|S = s]}$ <u>**Outside the Rasch model:**</u> <u>*What can go wrong?*</u>

$$\begin{aligned} P[X_j = 1|S = r-1] &= \int P[X_j = 1|S = r-1, \theta]\, f(\theta|S = r-1)d\theta \\ &\overset{?}{\leq} \int P[X_j = 1|S = r, \theta]\, f(\theta|S = r-1)d\theta \\ &\overset{?}{\leq} \int P[X_j = 1|S = r, \theta]\, f(\theta|S = r)d\theta \\ &= P[X_j = 1|S = r], \end{aligned}$$

- First "$\overset{?}{\leq}$" always true
- Second "$\overset{?}{\leq}$" fails when $P[X_j = 1|S = s, \theta]$ decreases in θ for fixed s.
- Failure must be large to offset first "$\overset{?}{\leq}$".

Note that our intuition based on item-total and item-rest correlations was backwards!

- *We expect* $Corr(X_j, S) > Corr(X_j, S_{(-j)})$ <u>*but*</u>
- $P[X_j = 1|S]$ *can be* <u>*nonmonotone*</u>*, whereas* $P[X_j = 1|S_{(-j)}]$ <u>*must*</u> <u>*be*</u> <u>*monotone*</u>*.*

Elementary Symmetric Functions

- Consider item subset $X_{k_1}, \ldots, X_{k_n}$ from $X_1, \ldots, X_J$

- Item Odds $\varepsilon_{k_j}(\theta) = P_{k_j}(\theta)/[1 - P_{k_j}(\theta)]$

- *Elementary symmetric function* for items $k_1, \ldots, k_n$, total score s, latent variable θ, and IRF's $\underline{P} = (P_1(\theta), \ldots, P_J(\theta))$, is

$$
\begin{aligned}
&\gamma_{(k_1,\ldots,k_n)}(s, \theta, \underline{P}) \\
&\quad = \sum_{s_{(k_1,\ldots,k_n)}=s} \prod_{j=1}^{n} \left[\frac{P_{k_j}(\theta)}{1 - P_{k_j}(\theta)}\right]^{x_{k_j}} \\
&\quad = \sum_{s_{(k_1,\ldots,k_n)}=s} \prod_{j=1}^{n} \varepsilon_{k_j}(\theta)^{x_{k_j}}
\end{aligned}
$$

  - Summation is over all score patterns such that $\sum_{j=1}^{n} X_{k_j} = s$

  - Also write $\gamma_{(k_1,\ldots,k_n)}(s, \theta, \underline{P}) \equiv \gamma_{(k_1,\ldots,k_n)}(s, \theta)$

Some properties of $\gamma_{(k_1,\ldots,k_n)}(s,\theta)$

- $$P[S=s \mid \theta] \;=\; \gamma_{(1,\ldots,J)}(s,\theta,\underline{P}) \cdot \prod_{j=1}^{J} (1-P_j(\theta))\ .$$

- Multiply by $f(\theta)$ and then divide by $P[S=s]$, get

$$\begin{aligned} f(\theta|S=s) &= \frac{P[S=s|\theta]f(\theta)}{P[S=s]} \\ &= \frac{\gamma_{(1,\ldots,J)}(s,\theta,\underline{P}) \cdot \prod_{j=1}^{J}(1-P_j(\theta))f(\theta)}{P[S=s]} \end{aligned}$$

- Integrate wrt. $f(\theta)$ and then multiply and divide by $P[S=0]$, get

$$P[S=s] \;=\; E[\gamma_{(1,\ldots,J)}(s,\Theta,\underline{P}) \mid S=0] \cdot P[S=0],$$

  a version of Holland's (1990) *Dutch identity*.

- Standard identities generalize from Rasch (e.g. Fischer 1974):

$$\begin{aligned} \gamma_{(1,\ldots,J)}(s,\theta,\underline{P}) &= \varepsilon_1(\theta)\gamma_{(2,\ldots,J)}(s-1,\theta,\underline{P}) \\ &\quad + \gamma_{(2,\ldots,J)}(s,\theta,\underline{P}) \\ \frac{d}{d\varepsilon_1(\theta)}\gamma_{(1,\ldots,J)}(s,\theta,\underline{P}) &= \gamma_{(2,\ldots,J)}(s-1,\theta,\underline{P}) \end{aligned}$$

Item-total regressions

Note that

$$
\begin{aligned}
P[X_1 = 1 \mid S = s, \theta] &= \frac{P[X_1 = 1, S = s \mid \theta]}{P[S_{(1,\ldots,J)} = s \mid \theta]} \\
&= \frac{P[X_1 = 1, S = s - 1 \mid \theta]}{P[S = s \mid \theta]} \\
&= \frac{\varepsilon_1(\theta)\gamma_{(2,\ldots,J)}(s-1, \theta, \underline{P}) \prod_{j=1}^{J}(1 - P_j(\theta))}{\gamma_{(1,\ldots,J)}(s, \theta, \underline{P}) \prod_{j=1}^{J}(1 - P_j(\theta))} \\
&= \frac{\varepsilon_1(\theta)\gamma_{(2,\ldots,J)}(s-1, \theta, \underline{P})}{\varepsilon_1(\theta)\gamma_{(2,\ldots,J)}(s-1, \theta, \underline{P}) + \gamma_{(2,\ldots,J)}(s, \theta, \underline{P})} \\
&= \left[1 + \frac{\gamma_{(2,\ldots,J)}(s, \theta, \underline{P})}{\varepsilon_1(\theta)\gamma_{(2,\ldots,J)}(s-1, \theta, \underline{P})}\right]^{-1}
\end{aligned}
$$

and hence

$$
P[X_1 = 1 \mid S = s, \theta] = \frac{1}{1 + \dfrac{\gamma_{(2,\ldots,J)}(s, \theta, \underline{P})}{\varepsilon_1(\theta)\gamma_{(2,\ldots,J)}(s-1, \theta, \underline{P})}} \qquad (3)
$$

Snijders' Example (Junker, 1993)

- The model:
  - Two-point distribution for Θ, $P(\Theta = \theta_0) = P(\Theta = \theta_1) = 1/2$, where $\theta_0 < \theta_1$.
  - $P_1(\theta_0) = \delta$, $P_1(\theta_1) = 1/2$
  - $P_j(\theta_0) = \delta$, $P_j(\theta_1) = 1 - \delta$ for $j = 2, 3, \ldots, J$.

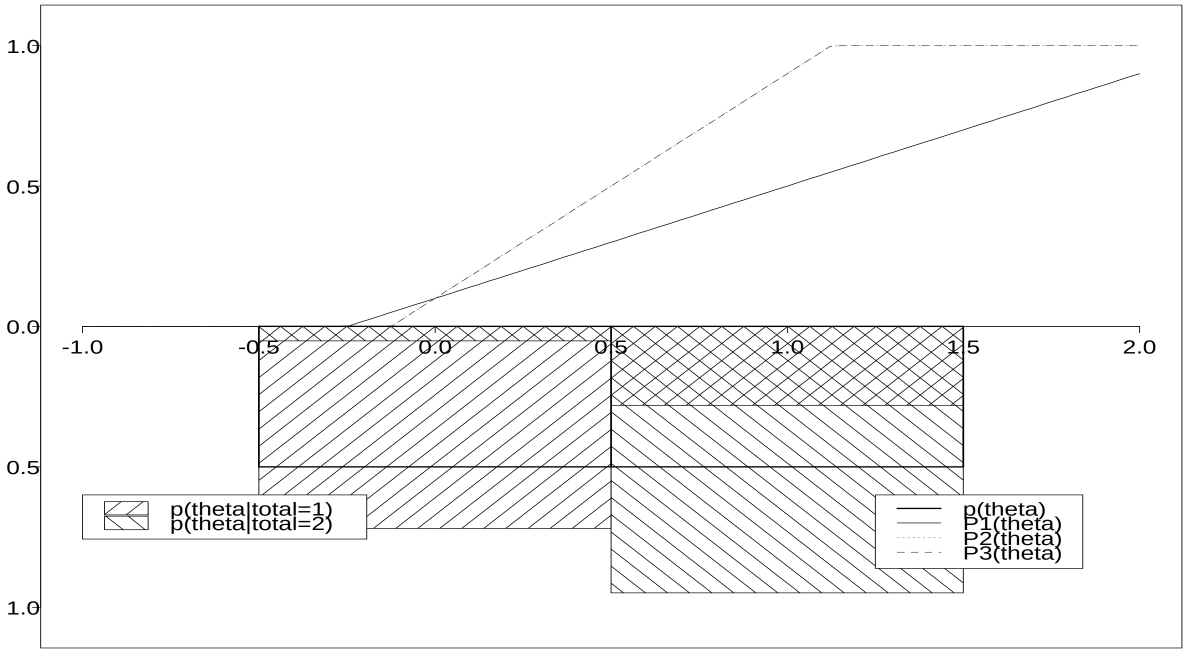
- For $J = 3$, as $\delta \to 0$, $P[X_1 = 1 | S = s]$ tends to the values 0, $\frac{1}{4}$, 0, and 1 for $s = 0, 1, 2, 3$, respectively.

- For all $J \geq 3$,

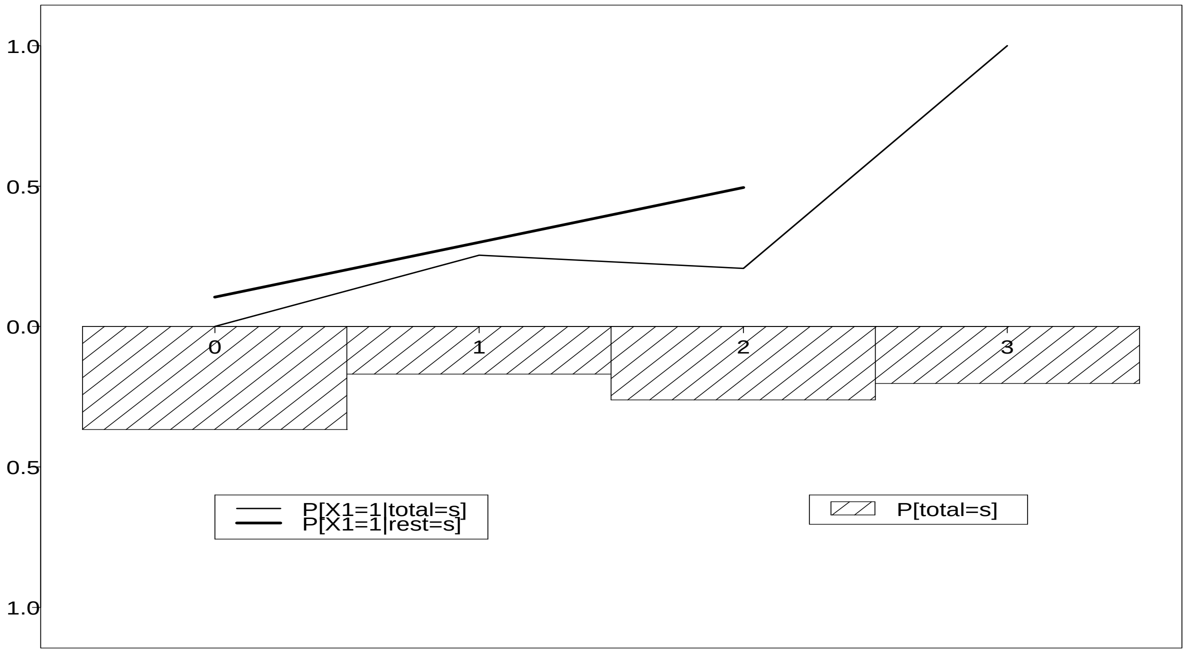
$$\begin{aligned} P[X_1 | S = s, \theta_0] &= \left[1 + \frac{J - s}{s}\right]^{-1} \\ P[X_1 | S = s, \theta_1] &= \left[1 + \frac{1-\delta}{\delta}\frac{J - s}{s}\right]^{-1} \end{aligned}$$

which is *decreasing* in θ, since $\frac{1-\delta}{\delta} > 1$ for $0 < \delta < 1$.

Snijders: Latent structure



Snijders: Manifest structure



Item-rest Corr = 0.39 Item-total Corr = 0.69

The 2PL model

- The model:
  - $f(\theta)$ = anything reasonable (e.g. std normal, or a bounded discrete approximation to the std normal).
  - $P_j(\theta) = \dfrac{e^{a_j(\theta-b_j)}}{1+e^{a_j(\theta-b_j)}}$, so $\varepsilon_j(\theta) = e^{a_j(\theta-b_j)} = e^{a_j\theta-\beta_j}$, where $\beta_j = a_j b_j$.

- Special case: $a_1 = a_1$ and $a_j = a_2$ for all $j = 2, \dots, J$.

$$\begin{aligned}
&P[X_1 = 1 \mid S = s, \theta] \\
&= \left[1 + \frac{\gamma_{(2,\dots,J)}(s,\theta,\underline{P})}{\varepsilon_1(\theta)\gamma_{(2,\dots,J)}(s-1,\theta,\underline{P})}\right]^{-1} \\
&= \left[1 + \frac{e^{sa_2\theta}\gamma_{(2,\dots,J)}(s,-,\underline{\beta})}{e^{a_1\theta-\beta_1}e^{(s-1)a_2\theta}\gamma_{(2,\dots,J)}(s-1,-,\underline{\beta})}\right]^{-1} \\
&= \left[1 + e^{(a_2-a_1)\theta}e^{\beta_1}\frac{\gamma_{(2,\dots,J)}(s,-,\underline{\beta})}{\gamma_{(2,\dots,J)}(s-1,-,\underline{\beta})}\right]^{-1},
\end{aligned}$$

where

$$\gamma_{(2,\dots,J)}(s,-,\underline{\beta}) = \sum_{s_{(-1)}=s} \prod_{j=2}^{n} e^{-\beta_j x_j}$$

- $a_1 = a_2$ (Rasch) $\Longrightarrow P[X_1 = 1 \mid S = s, \theta]$ <u>*const.*</u> in θ.

- $a_1 < a_2 \Longrightarrow P[X_1 = 1 \mid S = s, \theta]$ <u>*decreasing*</u> in θ

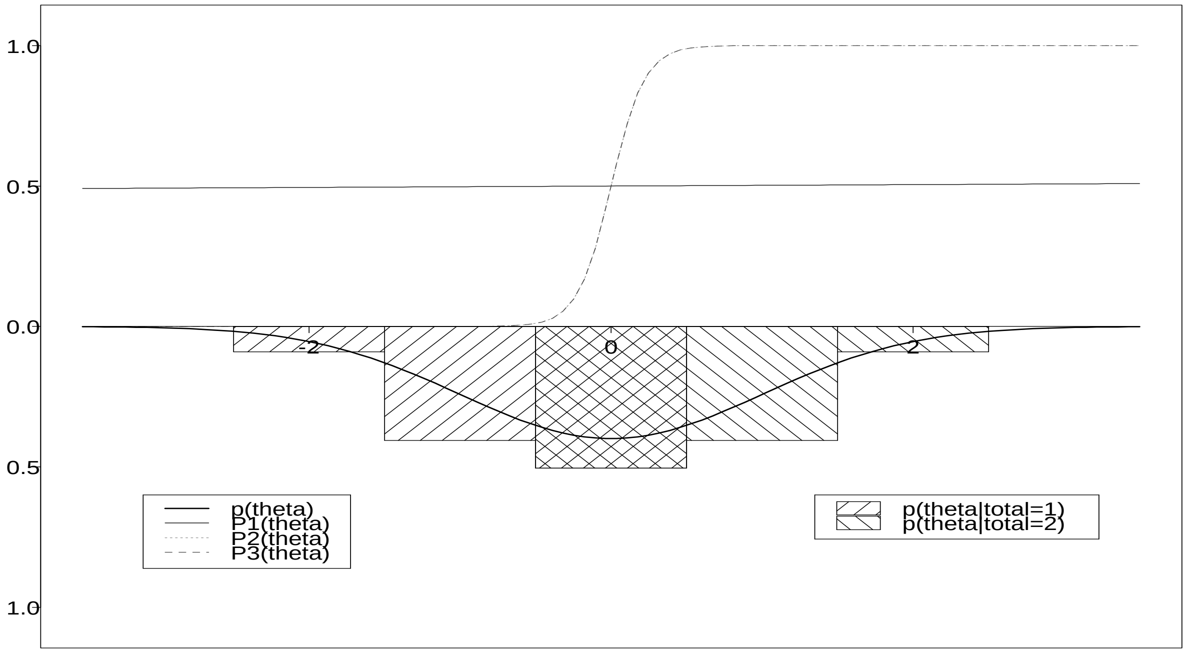
An extreme 2PL example

- $f(\theta)$ is symmetric, unimodal on $-2, -1, 0, 1, 2$.

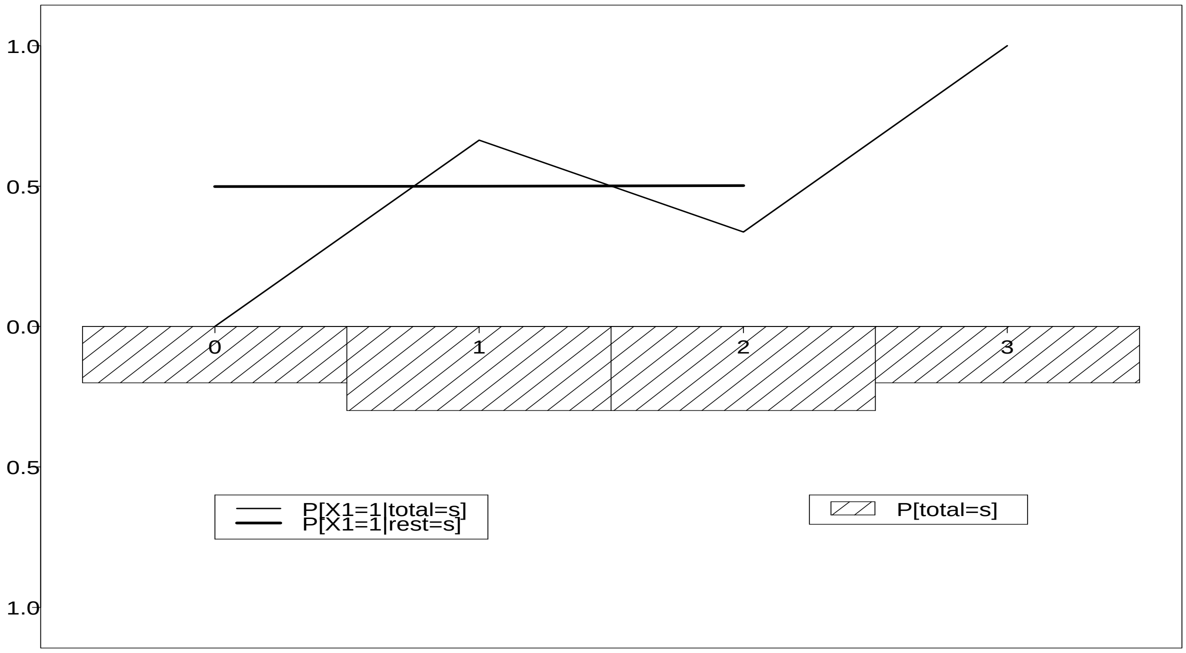
- $j = 1$: $P_1(\theta) \equiv 1/2$

- $j > 1$: $P_j(\theta) = \dfrac{e^{a\theta}}{1 + e^{a\theta}}$ where $a = 9$

Extreme 2PL: Latent structure

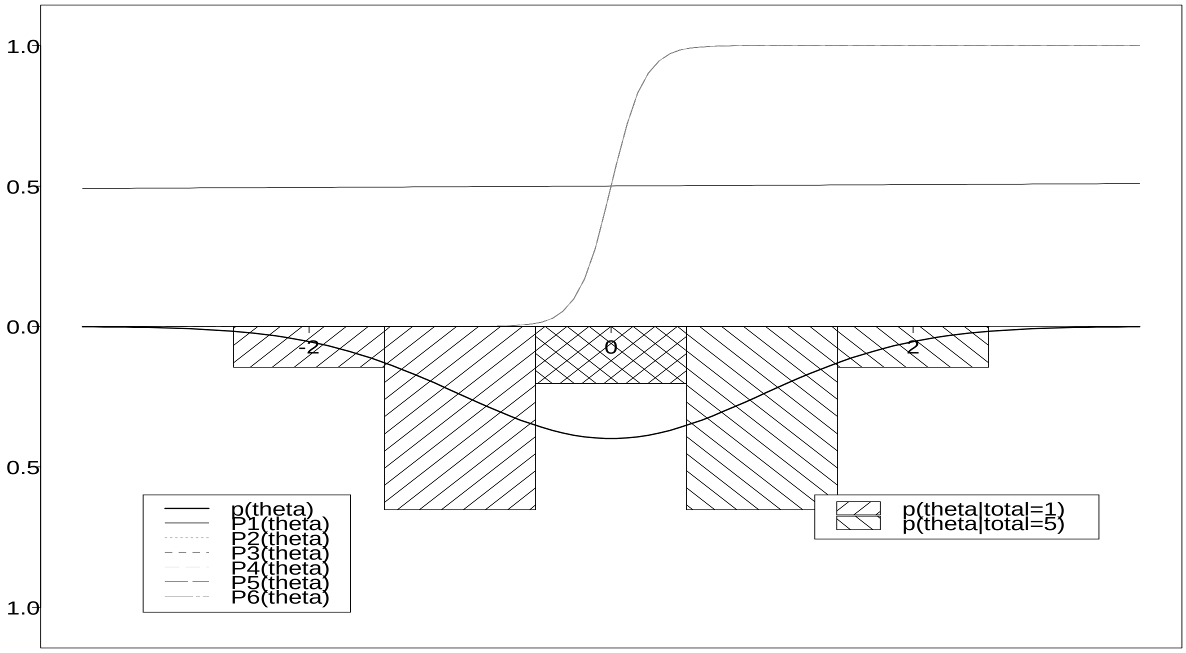


Extreme 2PL: Manifest structure

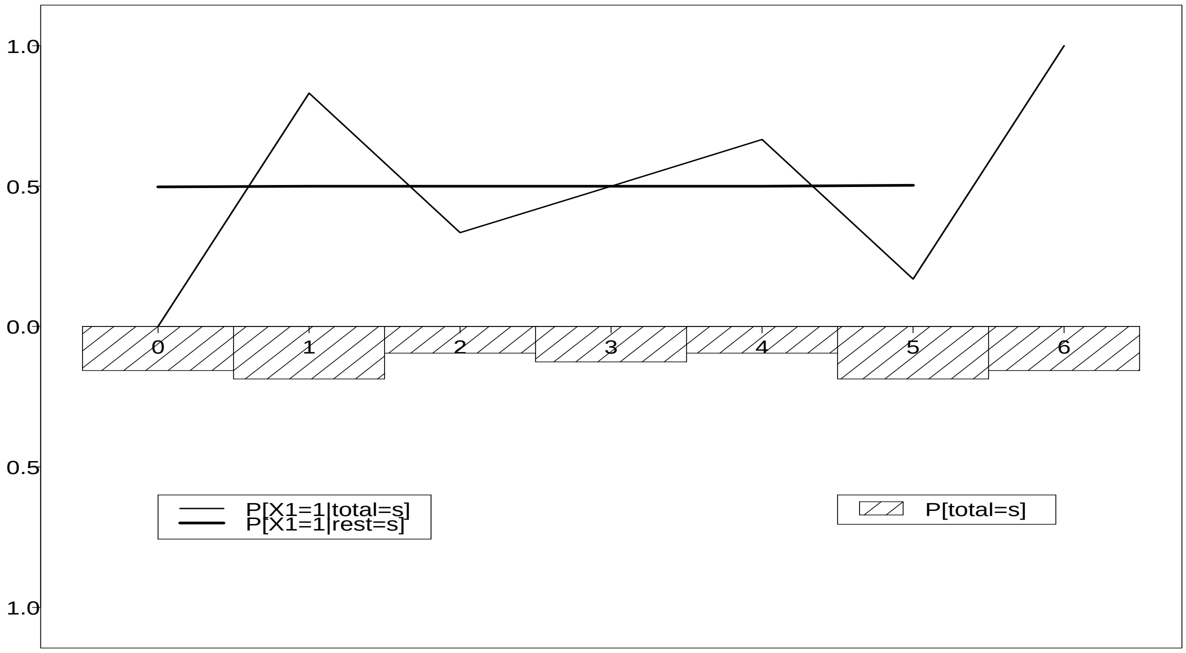


Item-rest Corr = 0 Item-total Corr = 0.49

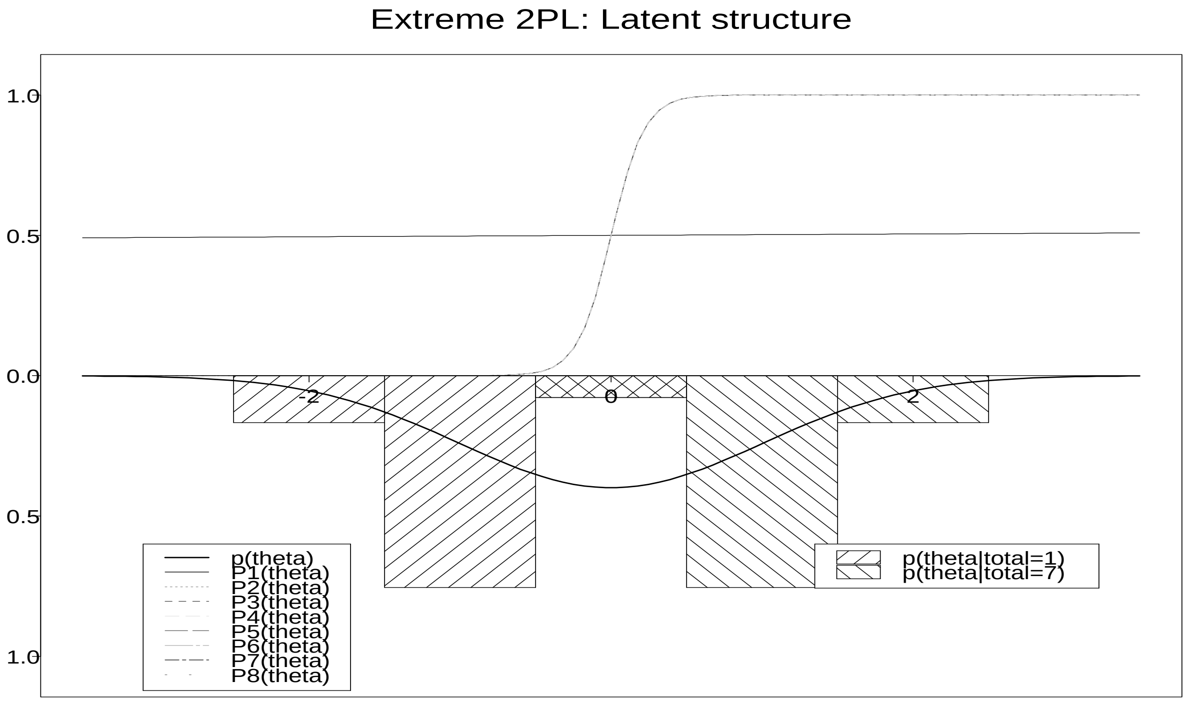
Extreme 2PL: Latent structure

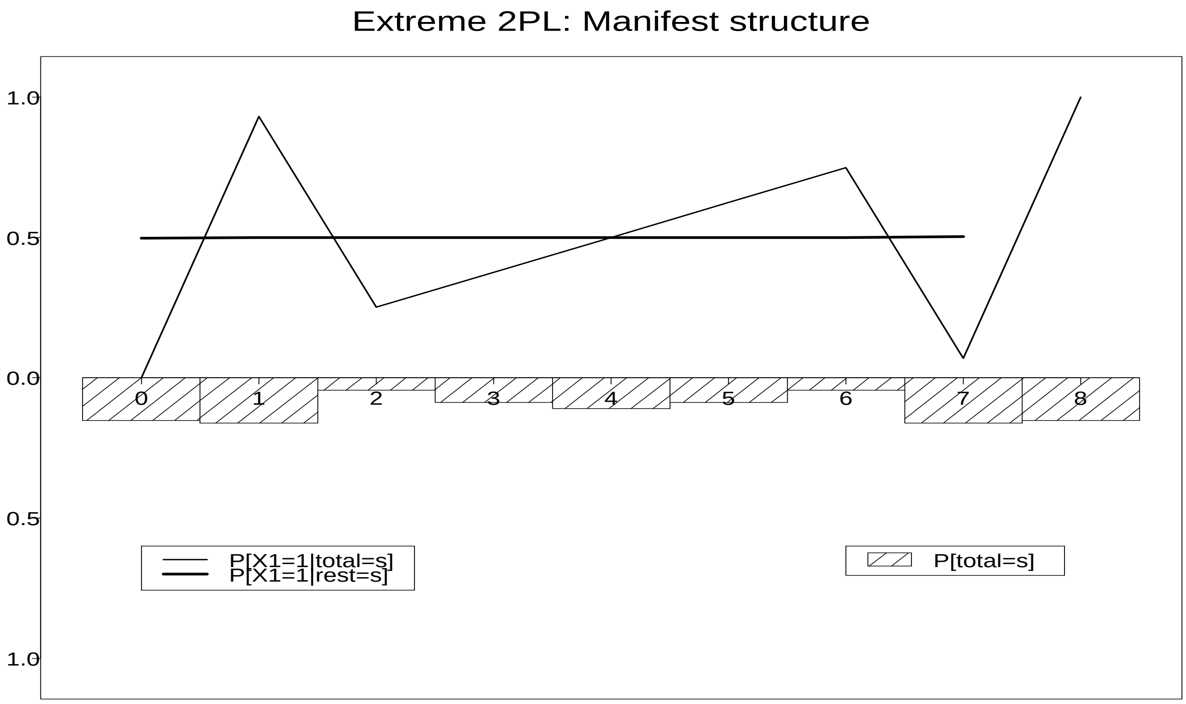


Extreme 2PL: Manifest structure

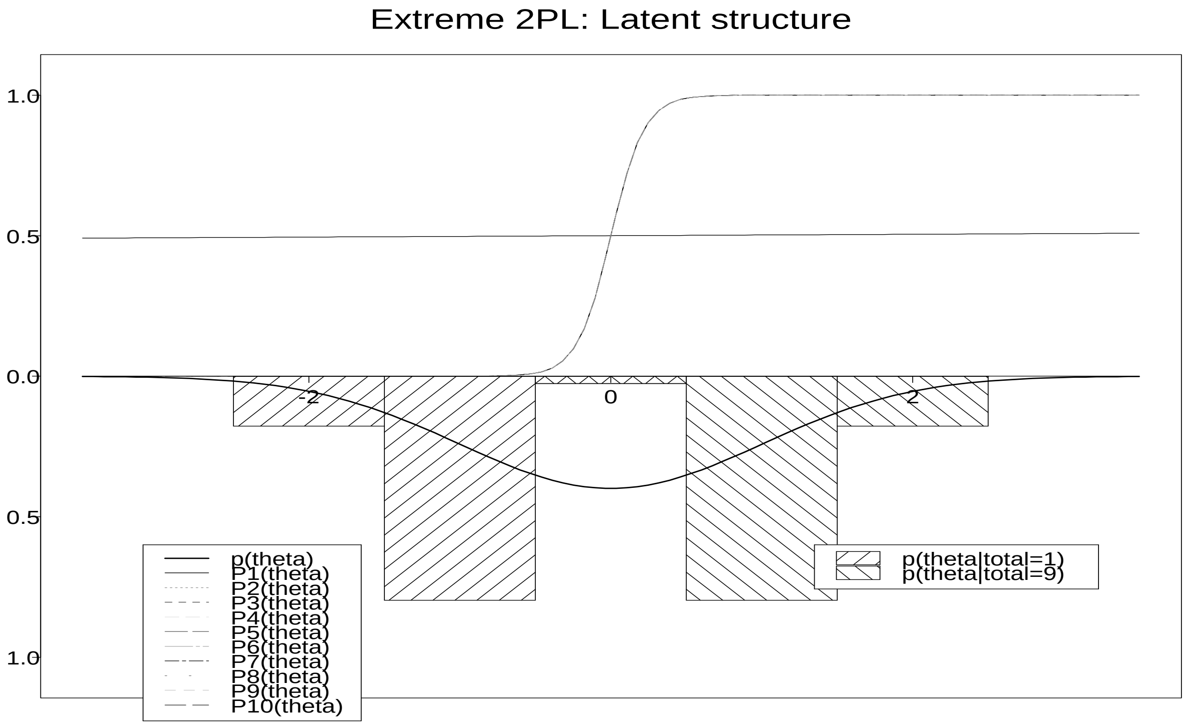


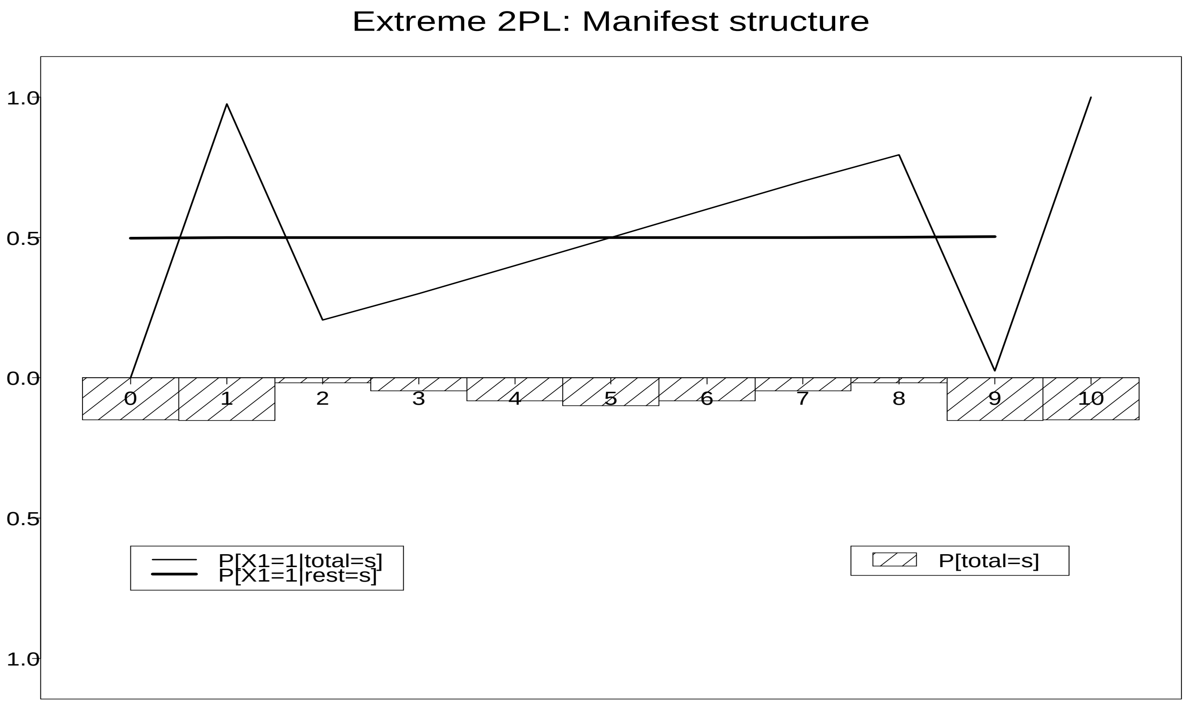
Item-rest Corr = 0 Item-total Corr = 0.24





Item-rest Corr = 0 Item-total Corr = 0.18



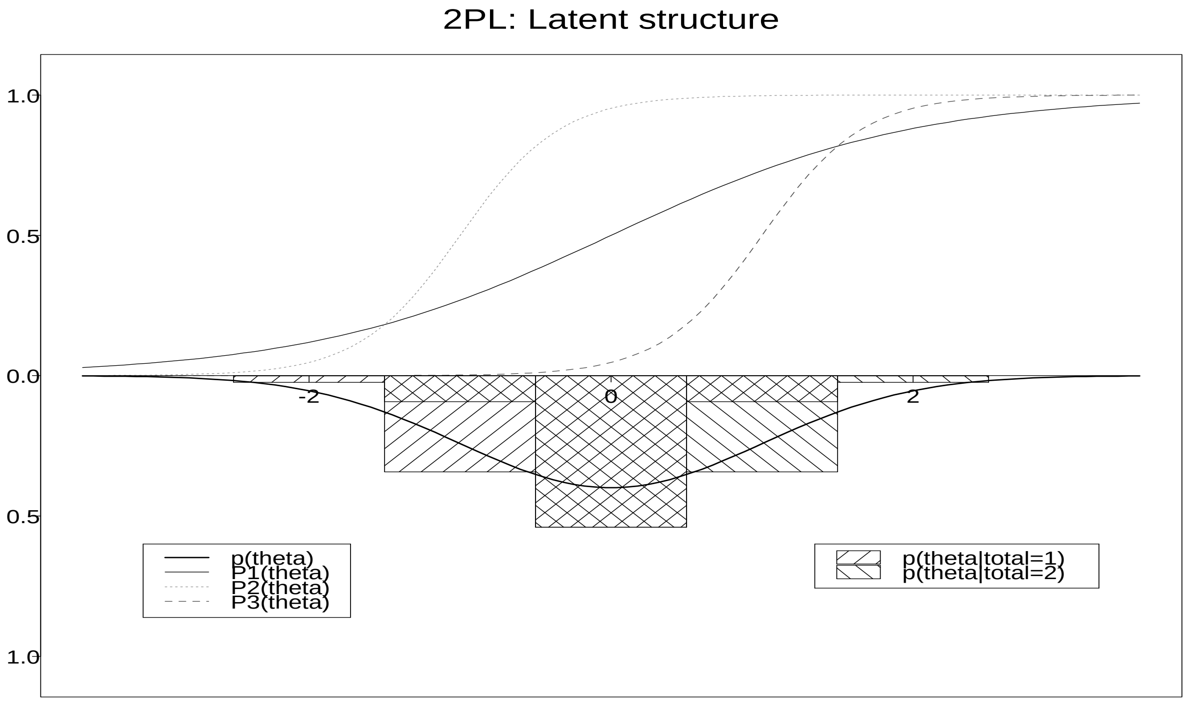


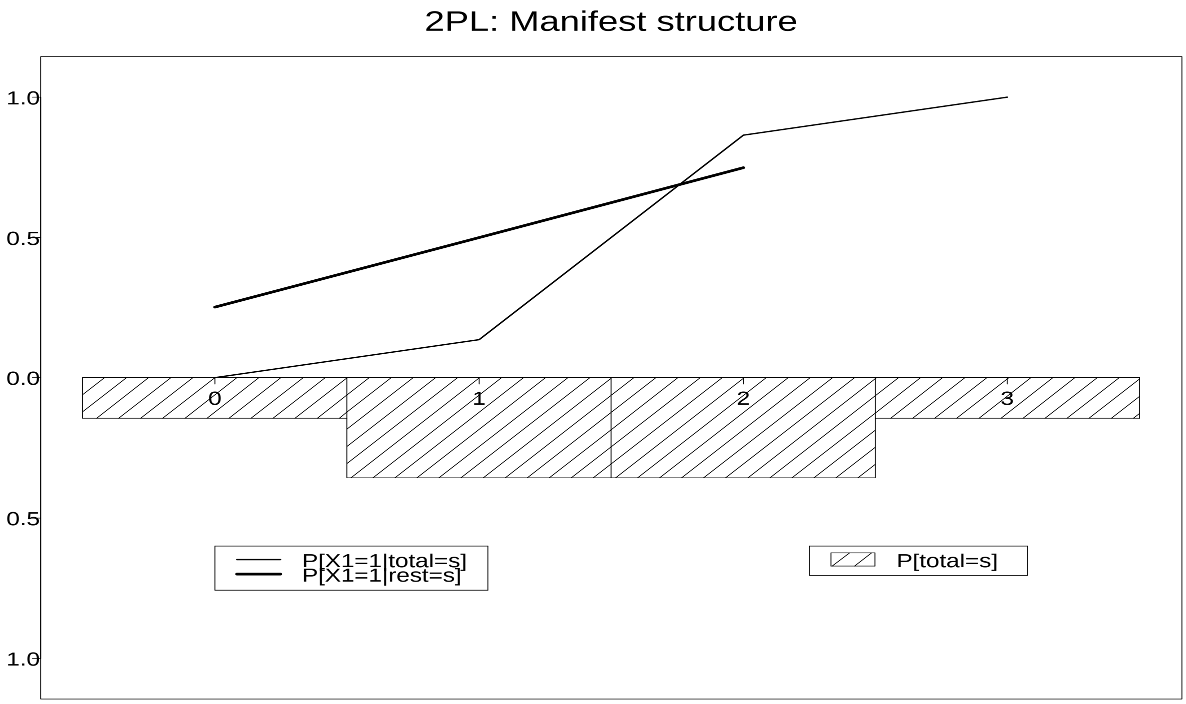
Item-rest Corr = 0 Item-total Corr = 0.14

2PL with varying difficulties

- $f(\theta)$ is symmetric, unimodal on -2, -1, 0, 1, 2.
- $j = 1$: $P_1(\theta)\dfrac{e^{\theta}}{1+e^{\theta}}$
- $j > 1$: $P_j(\theta) = \dfrac{e^{a(\theta-b_j)}}{1+e^{a(\theta-b_j)}}$,

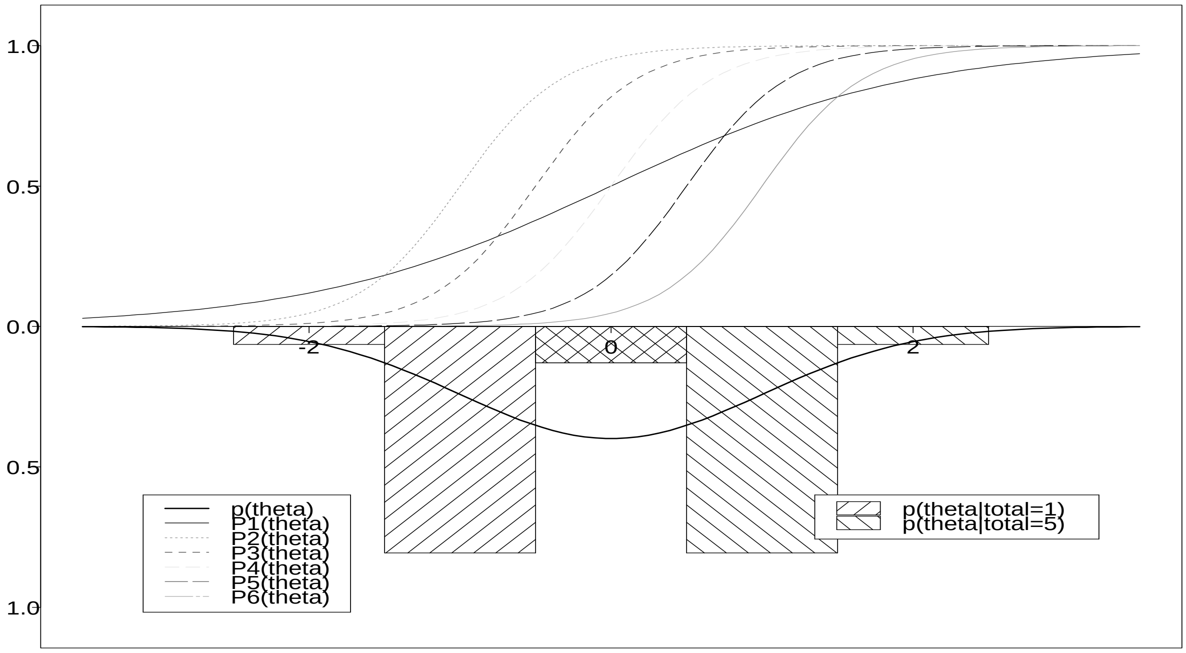
  where $a = 3$ and b_j equally spaced from -1 to $+1$.



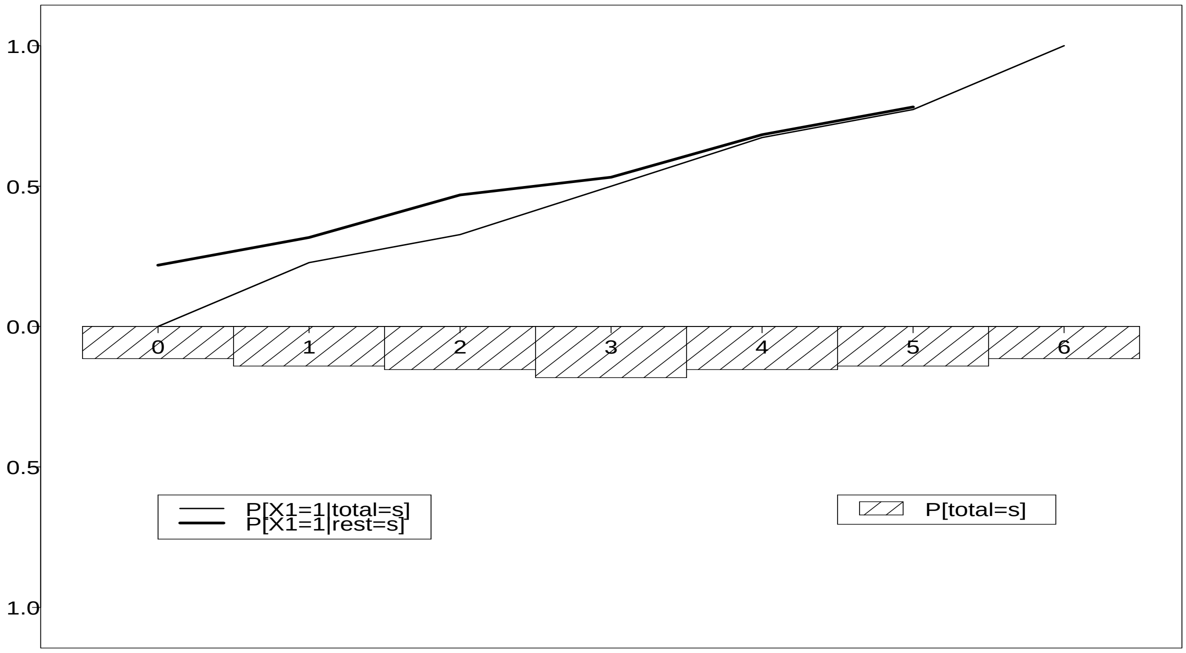


Item-rest Corr = 0.31 Item-total Corr = 0.76

2PL: Latent structure

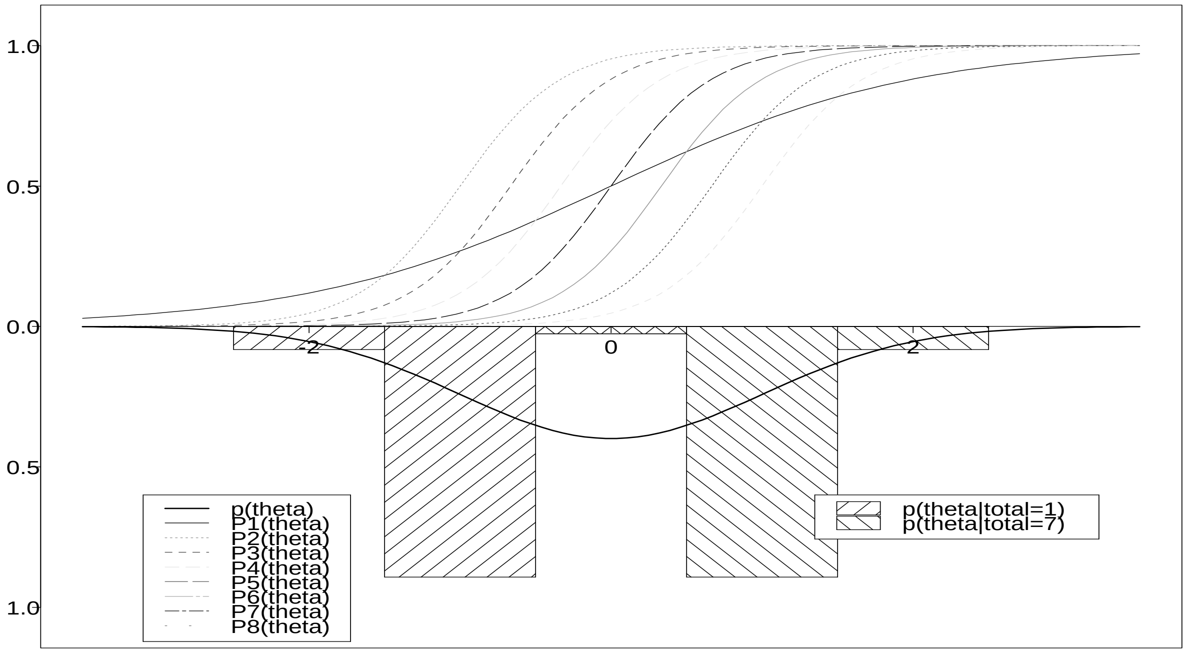


2PL: Manifest structure

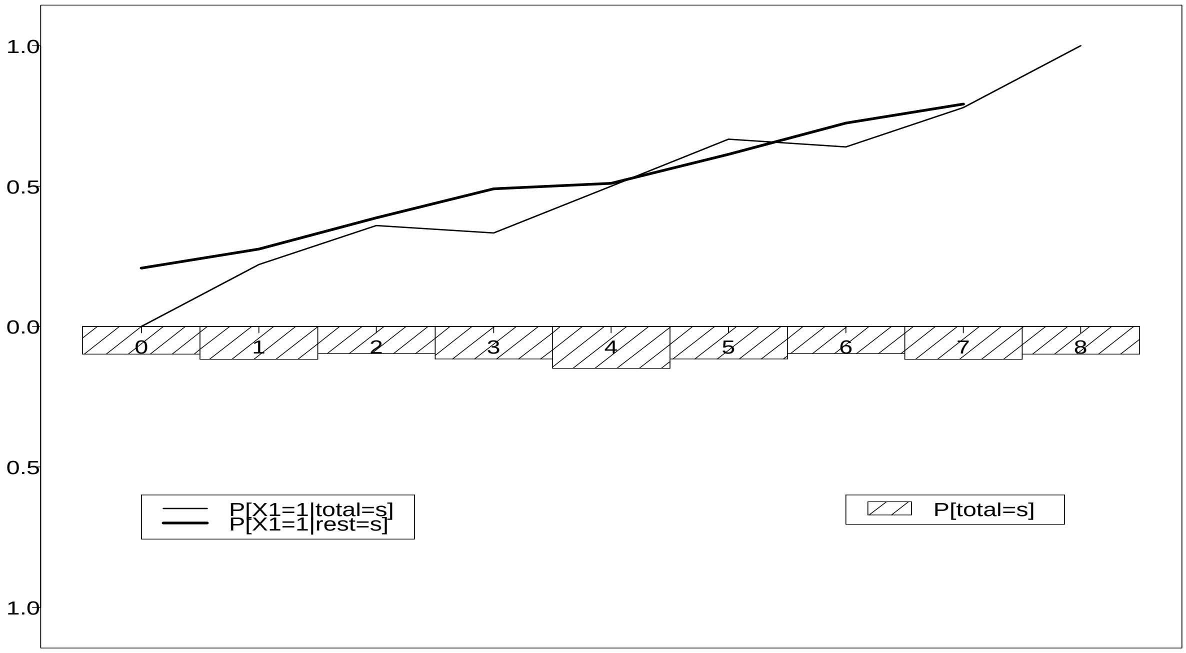


Item-rest Corr = 0.37 Item-total Corr = 0.59

2PL: Latent structure

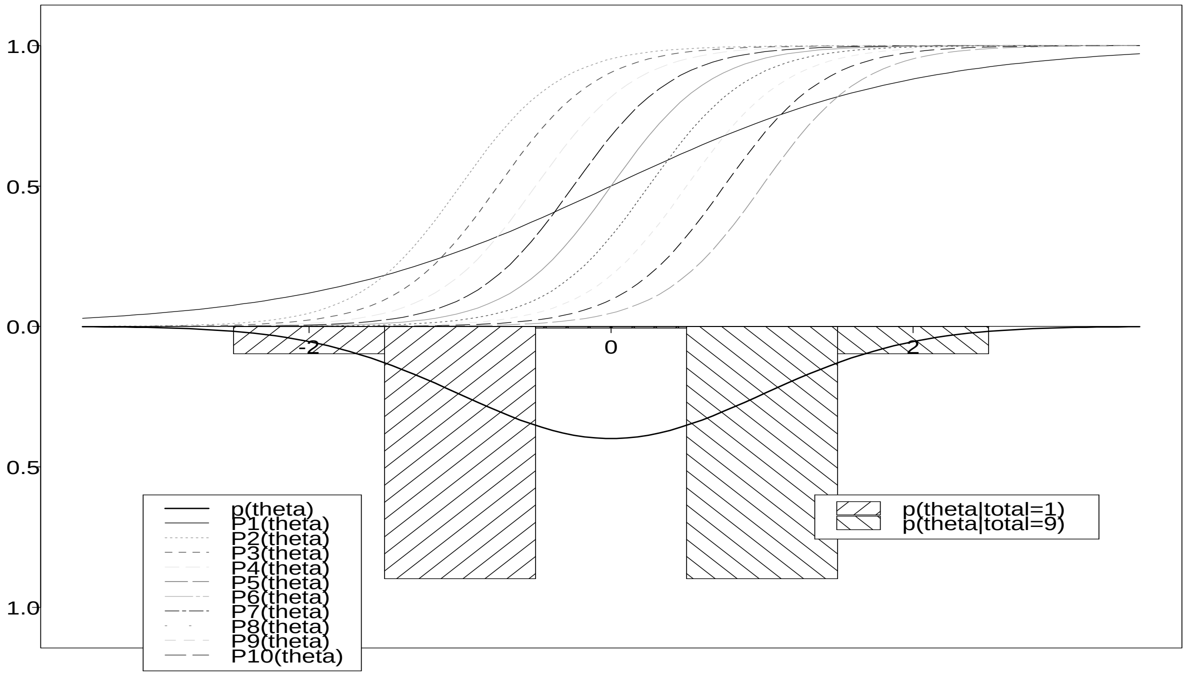


2PL: Manifest structure

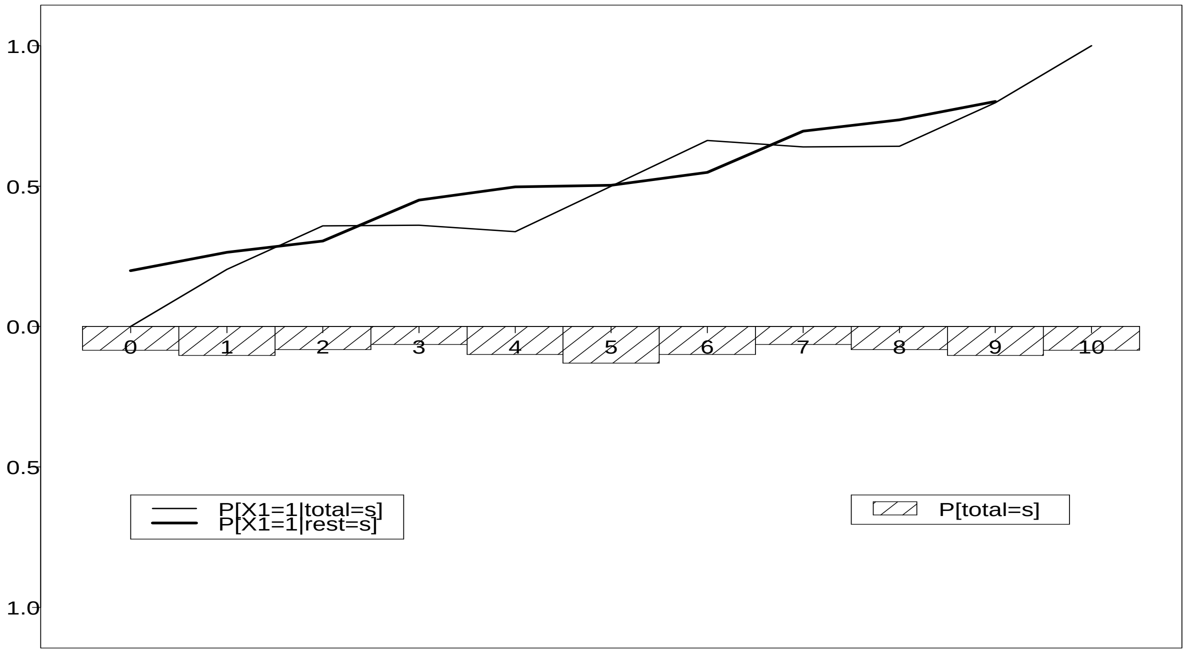


Item-rest Corr = 0.38 Item-total Corr = 0.54

2PL: Latent structure



2PL: Manifest structure

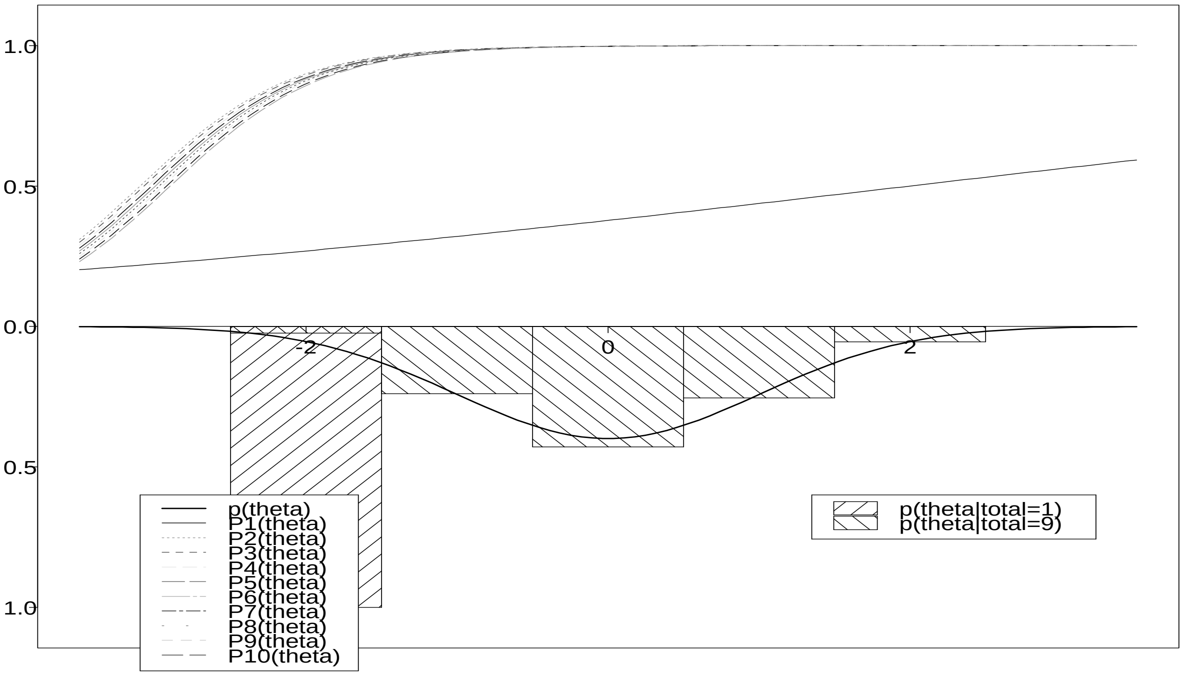


Item-rest Corr = 0.38 Item-total Corr = 0.52

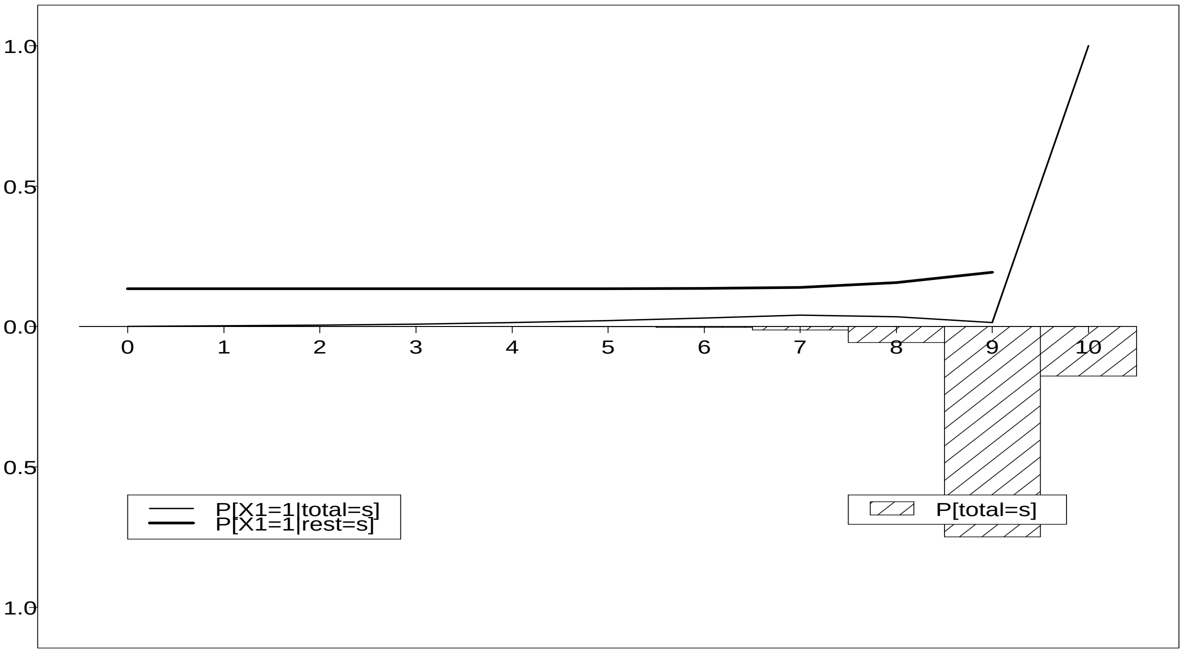
Noncrossing IRF's

- $f(\theta)$ is symmetric, unimodal on -2, -1, 0, 1, 2.
- $P_j(\theta) = \dfrac{e^{a_j(\theta - b_j)}}{1 + e^{a_j(\theta - b_j)}}$
- $j = 1$: $a_1 = 0.25$, $b_1 = 2$
- $j > 1$: $a_j = 2$, b_j equally spaced between -3.1 and -2.9
- Nonintersecting over the (finite) range of θ

IIO: Latent structure



IIO: Manifest structure



Item-rest Corr = 0.03 Item-total Corr = 0.71

Conclusions

Under the monotone unidimensional IRT model for binary items…

- $P[X_j = 1|S_{(-j)} = s]$ is always monotone in s

- $P[X_j = 1|S = s]$ monotone in the Rasch model

- Outside the Rasch model $P[X_j = 1|S = s]$ need not be monotone
  - Snijders' counter-example
  - Easy to construct 2PL and related counter-examples

- The nonmonotonicities clearly can get worse as J grows.

- We are now trying to generalize our examples, to produce some general criteria for non-monotonicity of $P[X_1 = 1 \mid S = s]$.