Information Theory

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Complex Systems Summer School

Entropy and Information Measuring randomness and dependence in bits

Entropy and Ergodicity Dynamical systems as information sources, long-run randomness

Information and Inference The connection to statistics

Cover and Thomas (1991) is the best single book on information theory.

Entropy

The most fundamental notion in information theory
X = a discrete random variable, values from $\mathcal{X}$
The **entropy of** X is

$$H[X] \equiv -\sum_{x \in \mathcal{X}} \Pr(X = x) \log_2 \Pr(X = x)$$

Proposition

$H[X] \geq 0$, *and* $= 0$ *only when* $\Pr(X = x) = 1$ *for some* x

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Proposition

$H[f(X)] \leq H[X]$, equality if and only if f is 1-1

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but the more fundamental interpretation is **description length**

Description Length

$H[X] =$ how concisely can we describe X?
Imagine X as text message:

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in Reno send lawyers guns and money kthxbai

Known and finite number of possible messages ($\#\mathcal{X}$)
I know what X is but won't show it to you
You can guess it by asking yes/no (binary) questions

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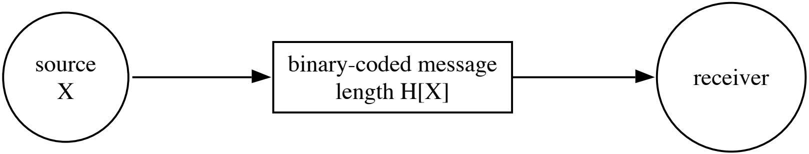
Mean is then $H[X]$

Theorem

$H[X]$ is the minimum mean number of binary distinctions needed to describe X

Units of $H[X]$ are **bits**



Multiple Variables — Joint Entropy

Joint entropy of two variables X and Y:

$$H[X,Y] \equiv -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \Pr(X=x, Y=y) \log_2 \Pr(X=x, Y=y)$$

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$$\begin{aligned} H[X,Y] &\geq H[X] \\ H[X,Y] &\geq H[Y] \\ H[X,Y] &\leq H[X] + H[Y] \\ H[f(X),X] &= H[X] \end{aligned}$$

Conditional Entropy

Entropy of conditional distribution:

$$H[X|Y=y] \equiv -\sum_{x \in \mathcal{X}} \Pr(X=x|Y=y) \log_2 \Pr(X=x|Y=y)$$

Average over y:

$$H[X|Y] \equiv \sum_{y \in \mathcal{Y}} \Pr(Y=y) H[X|Y=y]$$

On average, how many bits are needed to describe X, *after* Y is given?

$$H[X|Y] = H[X,Y] - H[Y]$$

"text completion" principle
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Chain rule:

$$H[X_1^n] = H[X_1] + \sum_{t=1}^{n-1} H[X_{t+1}|X_1^t]$$

Describe one variable, then describe 2nd with 1st, 3rd with first two, etc.

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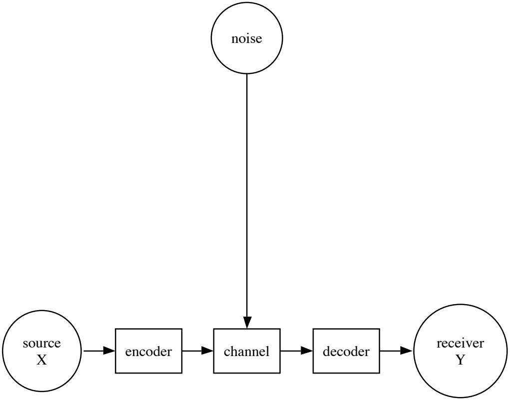
$$0 \leq I[X;Y] \leq \min H[X], H[Y]$$

$I[X;Y] = 0$ if and only if X and Y are statistically independent



How much can we learn about what was sent from what we receive? $I[X;Y]$

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This is *not* the only model of communication! (Sperber and Wilson, 1995, 1990)

Conditional Mutual Information

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Markov property is completely equivalent to

$$I[X_{t+1}^{\infty}; X_{-\infty}^{t-1}|X_t] = 0$$

Markov property is really about information flow

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Joint and conditional entropy definitions carry over
Mutual information definition carries over
MI *is* non-negative and invariant under 1-1 maps

Relative Entropy

$P, Q =$ two distributions on the same space $\mathcal{X}$

$$D(P\|Q) \equiv \sum_{x \in \mathcal{X}} P(x) \log_2 \frac{P(x)}{Q(x)}$$

Or, if $\mathcal{X}$ is continuous,

$$D(P\|Q) \equiv \int_{\mathcal{X}} dx\, p(x) \log_2 \frac{p(x)}{q(x)}$$

Or, if you like measure theory,

$$D(P\|Q) \equiv \int dP(\omega) \log_2 \frac{dP}{dQ}(\omega)$$

a.k.a. **Kullback-Leibler divergence**

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Invariant under 1-1 maps

Joint and Conditional Relative Entropies

P, Q now distributions on $\mathcal{X}, \mathcal{Y}$

$$D(P\|Q) = D(P(X)\|Q(X)) + D(P(Y|X)\|Q(Y|X))$$

where

$$\begin{aligned} D(P(Y|X)\|Q(Y|X)) &= \sum_x P(x) D(P(Y|X=x)\|Q(Y|X=x)) \\ &= \sum_x P(x) \sum_y P(y|x) \log_2 \frac{P(y|x)}{Q(y|x)} \end{aligned}$$

and so on for more than two variables

Relative entropy can be the basic concept

$$H[X] = \log_2 m - D(P \| U)$$

where $m = \#\mathcal{X}$, $U =$ uniform dist on $\mathcal{X}$, $P =$ dist of X

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$$I[X;Y] = D(J\|P \otimes Q)$$

where $P =$ dist of X, $Q =$ dist of Y, $J =$ joint dist

Relative Entropy and Miscoding

Suppose real distribution is P but we think it's Q and we use that for coding

Our average code length (**cross-entropy**) is

$$-\sum_{x} P(x) \log_2 Q(x)$$

But the optimum code length is

$$-\sum_{x} P(x) \log_2 P(x)$$

Difference is relative entropy

Relative entropy is the extra description length from getting the distribution wrong

Basics: Summary

Entropy $=$ minimum mean description length; variability of the random quantity
Mutual information $=$ reduction in description length from using dependencies
Relative entropy $=$ excess description length from guessing the wrong distribution

Information Sources

$X_1, X_2, \ldots X_n, \ldots$ a sequence of random variables
$X_s^t = (X_s, X_{s+1}, \ldots X_{t-1}, X_t)$
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Sequence of messages

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Need not be from a communication channel
e.g., successive states of a dynamical system
or *coarse-grained* observations of the dynamics

Definition (Strict or Strong Stationarity)

for any $k > 0$, $T > 0$, for all $w \in \mathcal{X}^k$

$$\Pr\left(X_1^k = w\right) = \Pr\left(X_{1+T}^{k+T} = w\right)$$

i.e., the distribution is invariant over time

Law of large numbers for stationary sequences

Theorem (Ergodic Theorem)

If X is stationary, then the empirical distribution converges

$$\hat{P}_n \rightarrow \rho$$

for some limit ρ, and for all nice functions f

$$\frac{1}{n}\sum_{t=1}^{n} f(X_t) \rightarrow \mathbf{E}_{\rho}\left[f(X)\right]$$

but ρ may be random and depend on initial conditions
one ρ per attractor

Entropy Rate

Entropy rate, a.k.a. **Shannon entropy rate**, a.k.a. **metric entropy rate**

$$h_1 \equiv \lim_{n \to \infty} H[X_n | X_1^{n-1}]$$

How many extra bits to we need to describe the next observation (in the limit)?

Theorem

h_1 exists for any stationary process (and some others)

Examples of entropy rates

IID $H[X_n|X_1^{n-1}] = H[X_1] = h_1$

Markov $H[X_n|X_1^{n-1}] = H[X_n|X_{n-1}] = H[X_2|X_1] = h_1$

k^{th}-order Markov $h_1 = H[X_{k+1}|X_1^k]$

Using chain rule, can re-write h_1 as

$$h_1 = \lim_{n \to \infty} \frac{1}{n} H[X_1^n]$$

description length per unit time

Topological Entropy Rate

$W_n \equiv$ number of allowed words of length n
$\equiv \# \left\{ w \in \mathcal{X}^n : \Pr\left(X_1^n = w\right) > 0 \right\}$
$\log_2 W_n \equiv$ **topological entropy**
topological entropy rate

$$h_0 = \lim_{n \to \infty} \frac{1}{n} \log_2 W_n$$

$H[X_1^n] = \log_2 W_n$ if and only if each word is equally probable
Otherwise $H[X_1^n] < \log_2 W_n$

Metric vs. Topological Entropy Rates

$$\begin{aligned} h_0 &= \text{growth rate in \# allowed words, counting all equally} \\ h_1 &= \text{growth rate, counting more probable words more heavily} \end{aligned}$$

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2^{h_1} = *effective* # of choices of how to go on

KS Entropy Rate

h_1 = growth rate of mean description length of *trajectories*

Chaos needs $h_1 > 0$

Coarse-graining deterministic dynamics, each partition $\mathcal{B}$ has its own $h_1(\mathcal{B})$

Kolmogorov-Sinai (KS) entropy rate:

$$h_{KS} = \sup_{\mathcal{B}} h_1(\mathcal{B})$$

Theorem

If $\mathcal{G}$ is a generating partition, then $h_{KS} = h_1(\mathcal{G})$

h_{KS} is the *asymptotic randomness* of the dynamical system

or, the rate at which the symbol sequence provides *new information* about the initial condition

Entropy Rate and Lyapunov Exponents

In general (Ruelle's inequality),

$$h_{KS} \leq \sum_{i=1}^{d} \lambda_i \mathbf{1}_{x>0}(\lambda_i)$$

If the invariant measure is smooth, this is equality (Pesin's identity)

Asymptotic Equipartition Property

When n is large, for any word x_1^n, either

$$\Pr\left(X_1^n = x_1^n\right) \approx 2^{-nh_1}$$

or

$$\Pr\left(X_1^n = x_1^n\right) \approx 0$$

More exactly, it's almost certain that

$$-\frac{1}{n}\log \Pr\left(X_1^n\right) \rightarrow h_1$$

This is the **entropy ergodic theorem** or **Shannon-MacMillan-Breiman theorem**

Relative entropy version:

$$-\frac{1}{n}\log Q_\theta(X_1^n) \rightarrow h_1 + d(P\|Q_\theta)$$

where

$$d(P\|Q_\theta) = \lim_{n\rightarrow\infty}\frac{1}{n}D(P(X_1^n)\|Q_\theta(X_1^n))$$

Relative entropy AEP implies entropy AEP

Entropy and Ergodicity: Summary

h_1 is the growth rate of the entropy, or number of choices made in continuing the trajectory

Measures instability in dynamical systems

Typical sequences have probabilities shrinking at the entropy rate

Relative Entropy and Sampling; Large Deviations

$X_1, X_2, \ldots X_n$ all IID with distribution P
Empirical distribution $\equiv \hat{P}_n$
Law of large numbers (LLN): $\hat{P}_n \rightarrow P$

Relative Entropy and Sampling; Large Deviations

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Theorem (Sanov)

$$-\frac{1}{n}\log_2 \Pr\left(\hat{P}_n \in A\right) \rightarrow \underset{Q \in A}{\operatorname{argmin}}\, D(Q\|P)$$

or, for non-mathematicians,

$$\Pr\left(\hat{P}_n \approx Q\right) \approx 2^{-nD(Q\|P)}$$

Sanov's theorem is part of the general theory of **large deviations**:

$$\Pr(\text{fluctuations away from law of large numbers}) \rightarrow 0$$

exponentially in n

rate functon generally a relative entropy

More on large devations: Bucklew (1990); den Hollander (2000)
LDP explains statistical mechanics; see Touchette (2008), or talk to Eric Smith

Relative Entropy and Hypothesis Testing

Testing P vs. Q
Optimal error rate (chance of guessing Q when really P) goes like

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For dependent data, substitute sum conditional relative entropy rate for D
The bigger $D(Q\|P)$, the easier is to test which is right

Method of Maximum Likelihood

Fisher (1922)
Data $= X$ with true distribution $= P$
Model distributions $= Q_\theta$, $\theta =$ parameter
Look for the Q_θ which best describes the data
Likelihood at θ is probability of generating the data
$Q_\theta(x) \equiv \mathcal{L}(\theta)$
Estimate θ by maximizing likelihood, equivalently log-likelihood
$\mathcal{L}(\theta) \equiv \log Q_\theta(x)$

$$\widehat{\theta} \equiv \operatorname*{argmax}_{\theta} \mathcal{L}(\theta) = \operatorname*{argmax}_{\theta} \sum_{t=1}^{n} \log Q_\theta(x_t | x_1^{t-1})$$

Maximum likelihood and relative entropy

Suppose we want the Q_θ which will best describe *new* data
Optimal parameter value is

$$\theta^* = \operatorname*{argmin}_{\theta} D(P \| Q_\theta)$$

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If $P = Q_{\theta_0}$ for some θ_0, then $\theta^* = \theta_0$ (true parameter value)
Otherwise θ^* is the **pseudo-true** parameter value

$$\begin{aligned}
\theta^* &= \underset{\theta}{\operatorname{argmin}} \sum_x P(x) \log_2 \frac{P(x)}{Q_\theta(x)} \\
&= \underset{\theta}{\operatorname{argmin}} \sum_x P(x) \log_2 P(x) - P(x) \log_2 Q_\theta(x) \\
&= \underset{\theta}{\operatorname{argmin}} -H_P[X] - \sum_x P(x) \log_2 Q_\theta(x) \\
&= \underset{\theta}{\operatorname{argmin}} -\sum_x P(x) \log_2 Q_\theta(x) \\
&= \underset{\theta}{\operatorname{argmax}} \sum_x P(x) \log_2 Q_\theta(x)
\end{aligned}$$

This is the *expected log-likelihood*

We don't know P but we do have $\hat{P}_n$
For IID case

$$\begin{aligned}\hat{\theta} &= \underset{\theta}{\operatorname{argmax}} \sum_{t=1}^{n} \log Q_\theta(x_t) \\ &= \underset{\theta}{\operatorname{argmax}} \frac{1}{n} \sum_{t=1}^{n} \log_2 Q_\theta(x_t) \\ &= \underset{\theta}{\operatorname{argmax}} \sum_{x} \hat{P}_n(x) \log_2 Q_\theta(x)\end{aligned}$$

So $\hat{\theta}$ comes from approximating P by $\hat{P}_n$
$\hat{\theta} \rightarrow \theta^*$ because $\hat{P}_n \rightarrow P$

Non-IID case (e.g. Markov) similar, more notation

Relative Entropy and Log Likelihood

In general:

$$-H[X] - D(P\|Q) = \text{expected log-likelihood of } Q$$
$$-H[X] = \text{optimal expected log-likelihood (ideal model)}$$

Why Maximum Likelihood?

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1. The inherent compelling rightness of the optimization principle (a bad answer)
2. Generally **consistent**: $\widehat{\theta}$ converges on the optimal value (as we just saw)
3. Generally **efficient**: converges faster than other consistent estimators

(2) and (3) are really theorems of probability theory
let's look a bit more at efficiency

Fisher Information

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Fisher information matrix

$$F_{uv}(\theta_0) \equiv -\mathbf{E}_{\theta_0}\left[\left. \frac{\partial^2 \log Q_{\theta_0}(X)}{\partial\theta_u \partial\theta_v} \right|_{\theta=\theta_0} \right]$$

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$F \propto n$ (for IID, Markov, etc.)
Variance of $\hat{\theta} = F^{-1}$ (under some regularity conditions)

The Information Bound

Theorem (Cramér-Rao)

F^{-1} is the minimum variance for any unbiased estimator

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because uncertainty in $\hat{\theta}$ depends on curvature at maximum leads to a whole **information geometry**, with F as the metric tensor (Amari *et al.*, 1987; Kass and Vos, 1997; Kulhavý, 1996; Amari and Nagaoka, 1993/2000)

Relative Entropy and Fisher Information

$$
\begin{aligned}
F_{uv}(\theta_0) &\equiv -\mathbf{E}_{\theta_0}\left[\left.\frac{\partial^2 \log Q_{\theta_0}(X)}{\partial\theta_u \partial\theta_v}\right|_{\theta=\theta_0}\right] \\
&= \left.\frac{\partial^2}{\partial\theta_u \partial\theta_v} D(Q_{\theta_0} \| Q_\theta)\right|_{\theta=\theta_0}
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Fisher information is how quickly the relative entropy grows with small changes in parameters

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Fisher information is how quickly the relative entropy grows with small changes in parameters

$$D(\theta_0 \| \theta_0 + \epsilon) \approx \epsilon^T F \epsilon + O(\|\epsilon\|^3)$$

Intuition: “easy to estimate” = “easy to reject sub-optimal values”

Maximum Entropy: A Dead End

Given *constraints* on expectation values of functions
$\mathbf{E}[g_1(X)] = c_1, \mathbf{E}[g_2(X)] = c_2, \ldots \mathbf{E}[g_q(X)] = c_q$

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$$\tilde{P}_{ME} \equiv \underset{P}{\operatorname{argmax}}\, H[P] : \forall i,\ \mathbf{E}_P[g_i(X)] = c_i$$

$$= \underset{P}{\operatorname{argmax}}\, H[P] - \sum_{i=1}^{q} \lambda_i (\mathbf{E}_P[g_i(X)] - c_i)$$

with **Lagrange multipliers** λ_i chosen to enforce the constraints

Solution: Exponential Families

Generic solution:

$$P(x) = \frac{e^{-\sum_{i=1}^{q} \beta_i g_i(x)}}{\int dx e^{-\sum_{i=1}^{q} \beta_i g_i(x)}} = \frac{e^{-\sum_{i=1}^{q} \beta_i g_i(x)}}{Z(\beta_1, \beta_2, \ldots \beta_q)}$$

again β enforces constraints
Physics: **canonical ensemble** with extensive variables g_i and intensive variables β_i
Statistics: **exponential family** with sufficient statistics g_i and natural parameters β_i
If we take this family of distributions as basic, MLE is β such that $\mathbf{E}[g_i(X)] = g_i(x)$, i.e., mean = observed

Best discussion of the connection is still Mandelbrot (1962)

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Calculate sample statistics $g_i(x)$

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Update distributions under new data by minimizing relative entropy
Often said to be the “least biased” estimate of P, or the one which makes “fewest assumptions”

About MaxEnt

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2. Conditional large deviations principle (Csiszár, 1995): if $\hat{P}$ is constrained to lie in a convex set A, then

$$-\frac{1}{n}\log \Pr\left(\hat{P} \in B | \hat{P} \in A\right) \rightarrow \inf_{Q \in B \cap A} D(Q\|P) - D(Q\|A)$$

so $\hat{P}$ is exponentially close to $\operatorname{argmin}_{Q \in A} D(Q\|P)$

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so $\hat{P}$ is exponentially close to $\mathrm{argmin}_{Q \in A} D(Q \| P)$
but the conditional LDP doesn't always hold

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- Updating by minimizing relative entropy can disagree with Bayes's rule (Seidenfeld, 1979, 1987; Grünwald and Halpern, 2003) , *contra* claims by physicists
- The "constraint rule" is certainly not required by logic or probability (Seidenfeld, 1979, 1987; Uffink, 1995, 1996)
- MaxEnt (or MinRelEnt) is not the best rule for coming up with a prior distribution to use with Bayesian updating; all such rules suck (Kass and Wasserman, 1996)

Minimum Description Length Inference

Rissanen (1978, 1989)

Chose a model to concisely describe the data
maximum likelihood minimizes description length of the *data*
...but you need to describe the model as well!
Two-part MDL:

$$\begin{aligned}
\mathcal{D}_2(x, \theta, \Theta) &= -\log_2 Q_\theta(x) + C(\theta, \Theta) \\
\widehat{\theta}_{MDL} &= \operatorname*{argmin}_{\theta \in \Theta} \mathcal{D}_2(x, \theta, \Theta) \\
\mathcal{D}_2(x, \Theta) &= \mathcal{D}_2(x, \widehat{\theta}_{MDL}, \Theta)
\end{aligned}$$

where C is a **coding scheme** for the parameters

Must fix coding scheme before seeing the data (EXERCISE: why?)
By AEP

$$n^{-1}\mathcal{D}_2 \rightarrow h_1 + \operatorname*{argmin}_{\theta \in \Theta} d(P \| Q_\theta)$$

still for finite *n* the coding scheme matters
(One-part MDL exists but would take too long)

Why Use MDL?

1. The inherent compelling rightness of the optimization principle
2. Good properties: for reasonable sources, if the **parametric complexity**

$$\mathrm{COMP}(\Theta) = \log \sum_{w \in \mathcal{X}^n} \operatorname*{argmax}_{\theta \in \Theta} Q_\theta(w)$$

   is small — if there aren't all that many words which get high likelihoods — then if MDL did well in-sample, it will generalize well to new data from the same source

See Grünwald (2005, 2007) for much more

Information and Statistics: Summary

Relative entropy controls large deviations
Relative entropy $=$ ease of discriminating distributions
Easy discrimination $\Rightarrow$ good estimation
Large deviations explains why MaxEnt works when it does

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