Quantitative Complexity Measures

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Complexity Measures

"Complex" $\approx$ "many strongly interacting *effective* degrees of freedom"

So not: only a few variables; most independent variables; lots of variables but only a few are relevant

Can we quantify this idea?

If so, what is the number good for?

I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of science, whatever the matter may be. — Thermodynamicist W. Thomson, a.k.a. Lord Kelvin

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but quantifying the wrong things advances a meagre and unsatisfactory understanding to the stage of pseudoscience, like IQ testing most complexity measures are “conspicuously vacuous” (Landauer, 1988) most complexity measures are “conspicuously vacuous” (Landauer, 1988)

> *The urge to destroy is also a creative urge.*
> *— Distributed systems theorist M. Bakunin*

Three Kinds of Complexity

1. *Description* of the system, in the preferred or optimal model (units: bits)
   Wiener, von Neumann, Kolmogorov, Pagels and Lloyd, ...
2. *Learning* that model (samples)
   Fisher, Neyman, Reichenbach, Vapnik and Chervonenkis, Valiant, ...
3. *Computational* complexity of the model (units: ops)

These are (pretty much) orthogonal
I will focus on description, with an occasional glance at learning

General references

Badii and Politi (1997)

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Feldman and Crutchfield (1998)

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Feldman and Crutchfield (1998)
Shalizi and Crutchfield (2001, appendices), Shalizi (2006, §8)
(discount appropriately)

What We Would Like

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High values for lots of strong interactions, lots of heterogeneity, lots of consequential options
Number should have implications about *other stuff*

Compression

Ordinary information theory: concise description of random objects

Can also think about coding and compression of particular objects, without reference to a generating distribution

Lossless compression: Encoded version is shorter than original, but can uniquely & exactly recover original

Lossy compression: Can only get something *close* to original

Stick with lossless compression

Compression by Programming

Lossless compression needs an **effective procedure** — definite steps which a machine could take to recover the original

Effective procedures = algorithms

Algorithms = recursive functions

Recursive functions = Turing machines

finite automaton with an unlimited external memory

Think about programs written in a universal language (R, Lisp, Fortran, C, C++, Pascal, Java, Perl, OCaml, Forth, ...)

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Length of this shortest program is $K_L(x)$

Why the big deal about L being universal?

1. Want to handle as general a situation as possible
2. Emulation: for any other universal language M, can write a compiler or translator from L to M, so

$$K_M(x) \leq |C_{L \to M}| + K_L(x)$$

Which universal language doesn't matter, much; and could use any other model of computation

Kolmogorov Complexity

The **Kolmogorov complexity** of x, relative to L, is

$$K_L(x) = \min_{p \in \mathcal{D}(x)} |p|$$

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$$1 \leq K_L(x) \leq |x| + c$$

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If $K_L(x) \approx |x|$, then x is **incompressible**

Examples

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π, first n digits: $K \leq g + \log_2 n$
In fact, any number you care to name contains little algorithmic information

Why?

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"uniform IID" = "pure noise" for short

Mean Algorithmic Information and Entropy Rate

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also (with more conditions) $n^{-1}K(X_1^n) \to h_1$ in probability

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2. Even if you could, it doesn't do what you want

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Use it to make a new program V which compresses the incompressible:

1. Sort all sequences by length and then alphabetically
2. For the i^{th} sequence $x^{(i)}$, use U to find $K_L(x^{(i)})$
3. If $K_L(x^{(i)}) \leq |V|$, keep going
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Due to Nohre (1994), cited by Rissanen (2003).

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Can never say: x is incompressible
Can say: haven't managed to compress x yet

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Power of this test is close to that of any other (computable) test (Martin-Lof)

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Makes $K_{L'}(x) = 1$, but makes descriptions of other strings longer

But the trick doesn't keep working

can translate between languages with constant complexity

still true that large incompressible sequences look like pure noise

ANY DETERMINISM DISTINGUISHABLE FROM RANDOMNESS IS INSUFFICIENTLY COMPLEX

Poincaré (2001) said as much 100 years ago, without the math

Excerpt on website

Extends to other, partially-compressible stochastic processes

The maximally-compressed description is incompressible

so other stochastic processes are transformations of noise

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For much better ideas on Occam's Razor, see `http://www.andrew.cmu.edu/user/kk3n/ockham/Ockham.html`

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Logical Depth

Bennett (1985, 1986, 1990)

Logical depth of $x \approx$ how long does the shortest program for x take to run?
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Best reference on Kolmogorov complexity: Li and Vitányi (1997)

Thermodynamic Depth

Lloyd and Pagels (1988)

Thermodynamic depth $=$ Shannon entropy of trajectories leading to the current state
How many bits do we need to describe the particular history that assembled this state (given that it did end up here)?
Simple states have easily-described histories
Complex states have histories that need lots of information
Alas: depth grows to infinity in a stationary process
See Crutchfield and Shalizi (1999)

Minimal Sufficient Statistics (encore)

Recall from last time:

- A statistic (function of the history) ϵ is **sufficient** when $I[X_{t+1}^{\infty}; X_{-\infty}^{t}] = I[X_{t+1}^{\infty}; \epsilon(X_{-\infty}^{t})]$
- A sufficient statistic is **minimal** when $\epsilon = g(\eta)$ for any other sufficient η, thus $I[X_{-\infty}^{t}; \epsilon(X_{-\infty}^{t})] \leq I[X_{-\infty}^{t}; \eta(X_{-\infty}^{t})]$
- Minimal sufficient statistics are unique (up to re-labeling of values)
- We can construct them and (sometimes) estimate them

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Split the difference and call it GCY complexity

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$=$ information about microstate in macroscopic observations (sometimes)

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a.k.a. effective measure complexity, excess entropy, ...
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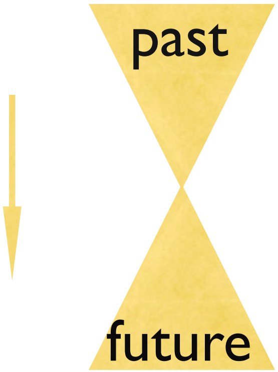
Efficiency of prediction = $I_{mathrmpred}/C_{GCY} \leq 1$

Spatio-Temporal Prediction

Dynamic random field: $X(\vec{r}, t)$
Assume a finite "speed of light"
Past light cone of $(\vec{r}, t)$: all points at earlier times from which a signal could have come
Future light cone: all points at later times to which a signal could go



Light cones in $1 + 1$D

Local Causal States

Go through equivalence classing again, only now for predicting the configuration in the future cone from that in the past cone
Still minimal sufficient statistics, recursive updating (on new information), local states form a Markov random field (Shalizi, 2003; Shalizi *et al.*, 2004, 2006)

Self-Organization

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Emergence

Start with a process (X_t) at one level of description, get $C(X)$, $I_{\mathrm{pred}}(X)$

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Can e.g. show that thermodynamic descriptions emerge from statistical-mechanical ones (Shalizi and Moore, 2003)

Local Statistical Complexity

Shalizi *et al.* (2006)

$$C(\vec{r}, t) \equiv -\log \Pr\left(S = s(\vec{r}, t)\right)$$

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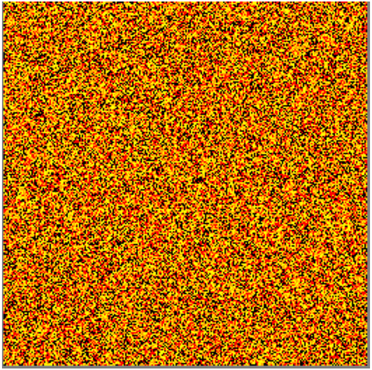
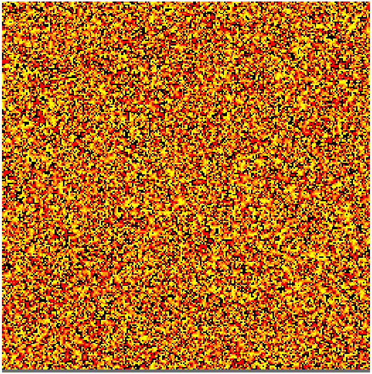
Can change over space and time
Use to automatically filter for the interesting bits

Cyclic Cellular Automata, as an Example

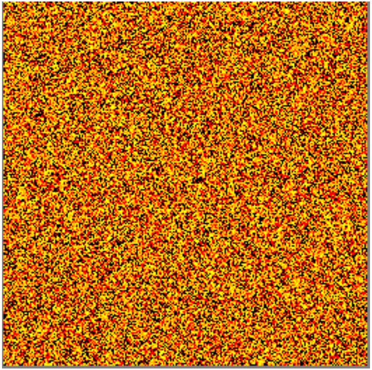
Quantitative model of excitable media
κ colors; a cell of color k switches to $k + 1 \mod \kappa$ if at least T neighbors are already of that color
Analytical theory for structures formed (Fisch *et al.*, 1991a,b)
Generic behaviors: spirals, "turbulence", local oscillations, fixation



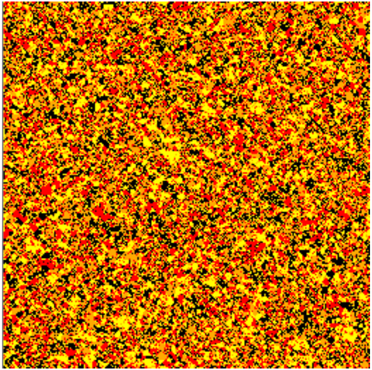
Initial configuration, $T = 1$



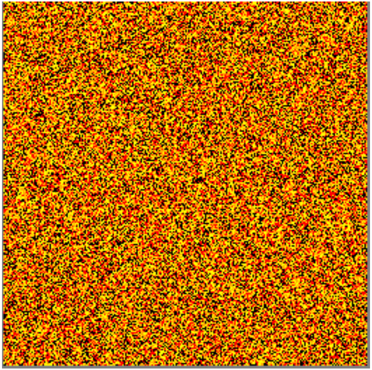
Final configuration, $T = 1$ (oscillates forever)



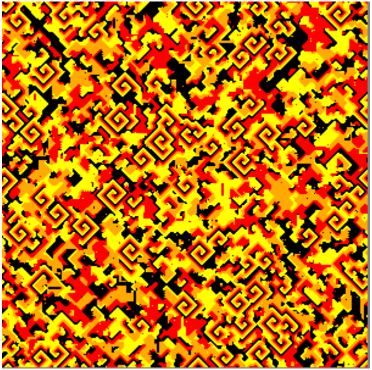
Initial configuration, $T = 4$



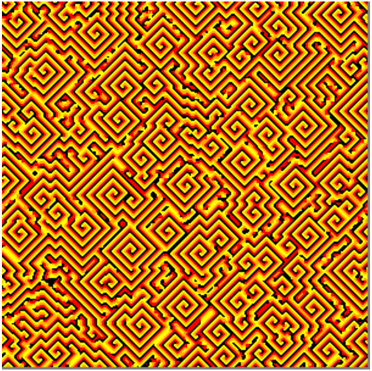
Final configuration, $T = 4$ (static blocks)



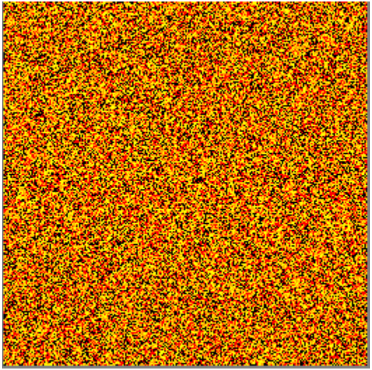
Initial configuration, $T = 2$



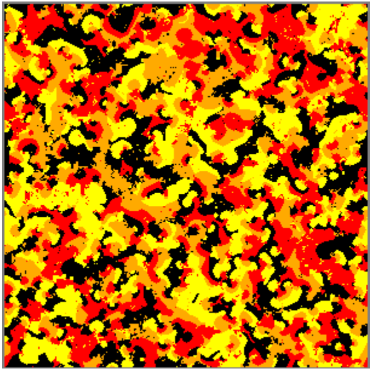
Intermediate time configuration, $T = 2$



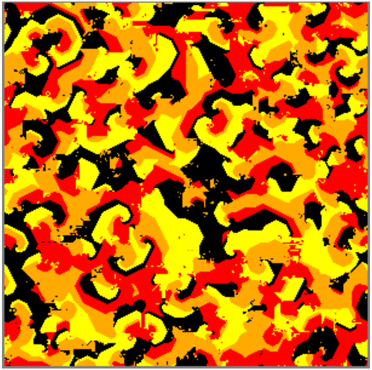
Asymptotic configuration, $T = 2$, rotating spirals



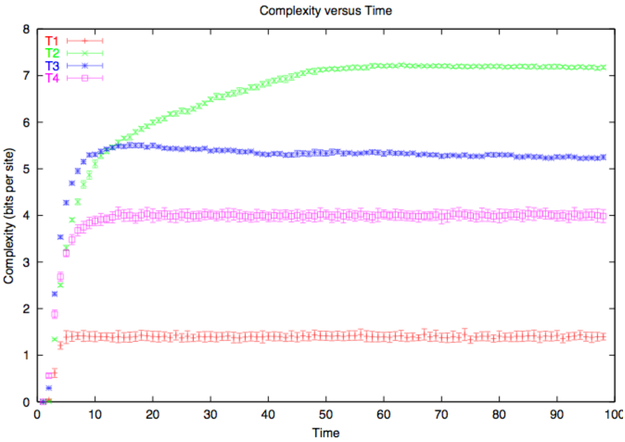
Initial configuration, $T = 3$



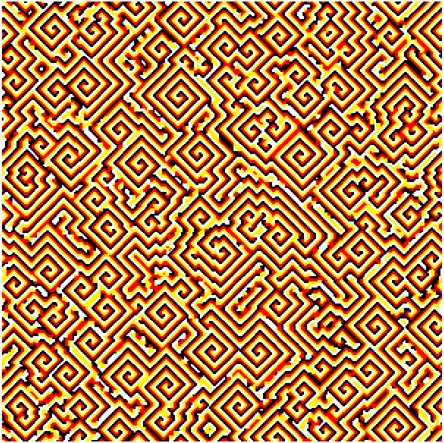
Intermediate time configuration, $T = 3$



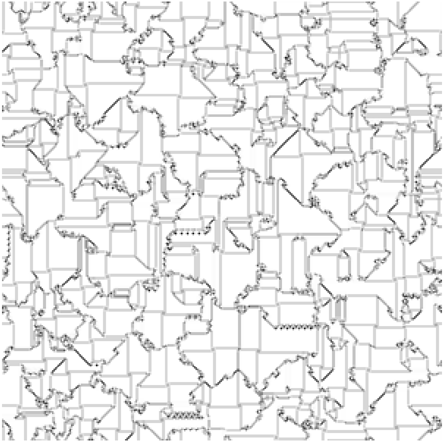
Asymptotic configuration, $T = 3$, turbulent seething gurp



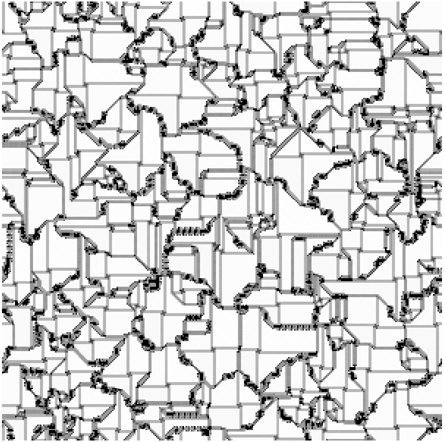
C_{GCY} vs. time and threshold, 300×300 lattice, 30 replicas



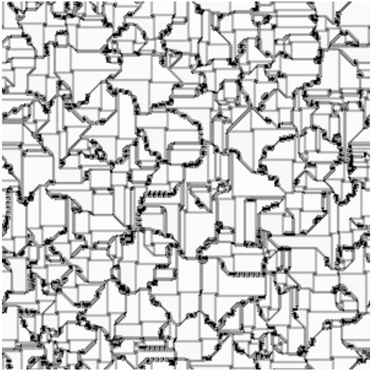
Typical long-time configuration



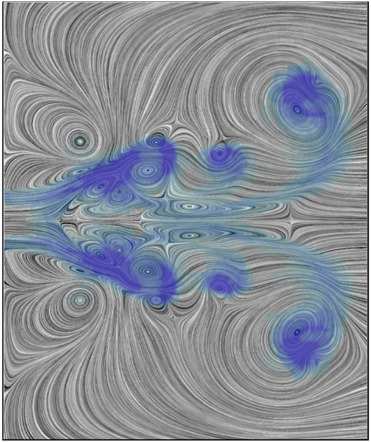
Hand-crafted order parameter field



Local complexity field



Order parameter (broken symmetry, physical insight, tradition, trial and error, current configuration) vs. local statistical complexity (prediction, automatic, time evolution)



Streamlines from computational fluid dynamics; color indicates local complexity of velocity field (Jänicke *et al.*, 2007)

Zombie Complexities

Ideas which should be dead, but continue to eat brains

- Prigogine's ideas on dissipative structures
- Haken's synergetics
- Wolfram's 4 classes of CA
- The edge of chaos — see Mitchell *et al.* (1993)
- $(\text{disorder}) \times (1 - \text{disorder})$ — see Binder and Perry (2000); Crutchfield *et al.* (2000)
- Self-organized criticality (as a ruling idea)
- Power-laws, *therefore* complex
- Tsallis statistics

Why Physicists Think Power Laws Are Cool

Go back to fundamental statistical mechanics
Macroscopic variable $M =$ coarse-graining of microscpic state
$W(m) =$ volume of microstates x such that $M(x) = m$
Boltzmann entropy $S_B(m) = \log W(m)$
Equilibrium = state m^* maximizing S_B
Einstein formula for fluctuations around equilibrium:

$$\Pr(M = m) \propto e^{S_B(m)}$$

Expand around m^*, so $\partial S_B / \partial m = 0$ at m^*

$$
\begin{aligned}
\Pr(M = m) &\propto e^{S(m^*) + \frac{1}{2}\frac{\partial^2 S(m^*)}{\partial m^2}(m - m^*)^2 + \mathrm{h.o.t.}} \\
&\propto e^{\frac{1}{2}\frac{\partial^2 S(m^*)}{\partial m^2}(m - m^*)^2 + \mathrm{h.o.t.}}
\end{aligned}
$$

drop the h.o.t.

$$
M \sim \mathcal{N}(m^*, -\frac{\partial^2 S(m^*)}{\partial m^2})
$$

What's Really Going On

correlations are short range
$\Rightarrow$ rapid approach to independence, exponential mixing
$\Rightarrow$ central limit theorem for averages over space (and time)
$\Rightarrow$ Gaussians for macroscopic variables (which are averages)

Phase Transitions

See Yeomans (1992) for nice introduction

Basically, bifurcations: behavior changes suddenly as temperature (or pressure or other control variable) crosses some threshold

First order: entropy is discontinuous at critical point

Examples: ice/water at 273K (and standard pressure); water/steam at 373K

order parameter is discontinuous

Second order: *derivative* of entropy is discontinuous

Example: “Curie point”, permanent magnetization/not in iron 1043K

order parameter continuous but with sharp kink

like amplitude of limit cycle in period-doubling

Focus on continuous (second-order) case

Critical fluctuations

Entropy story breaks down because derivatives $\rightarrow \pm\infty$
Competition between two phases, no preference, one can pop up in the middle of the other
Fluctuations get arbitrarily large $\Rightarrow$ long-range correlations $\Rightarrow$ slow mixing (if any)
Assemblage becomes self-similar: magnify a small part and it looks just like the whole thing ("renormalization")
only strictly true for infinitely big assemblages
averaging must lead to a self-similar distribution
Power laws are self-similar (scale-free)
Conclusion: at critical point, expect to see power law distributions

Theory of phase transitions / critical phenomena / order parameters / renormalization one of the key developments in physics over the last half century (Yeomans, 1992; Domb, 1996)

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(ii) is called "the fallacy of affirming the consequent"

Many ways to get power laws or other heavy-tailed distributions

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e.g., exponential growth for a random time (Reed and Hughes, 2002)

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e.g., exponential growth for a random time (Reed and Hughes, 2002)
or multiplicative fluctuations (Simon, 1955)

Tsallis Statistics

Take the MaxEnt procedure, but instead maximize

$$H_q[X] \equiv \frac{1}{q-1}\left(1 - \sum_x (\Pr(X = x))^q\right)$$

(similar form for continuous case)
Reverts to Shannon entropy as $q \to 1$
leads to "q-exponential" CDF

$$P_{q,\kappa}(X \geq x) = \left(1 - \frac{(1-q)x}{\kappa}\right)^{1/(1-q)}$$

q-Exponentials

(Shalizi, 2007) Set

$$q = 1 + \frac{1}{\theta}, \ \kappa = \frac{\sigma}{\theta}$$

Observe

$$P_{\theta,\sigma}(X \geq x) = (1 + x/\sigma)^{-\theta}$$

vs. "type II generalized Pareto distribution" (Arnold, 1983)

$$P(X \geq x) = [1 + (x - \mu)/\sigma]^{-\alpha}$$

set $\mu = 0$ and $\alpha = \theta$
Comes from a mixture of exponentials (Maguire *et al.*, 1952)

Tsallis statistics supposedly good for long-range interactions

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If you want more:
`http://bactra.org/notebooks/tsallis.html`

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