Stat 220 – Final Exam – December 16, 2004 – Schafer

NAME: ______________________________

ANDREW ID: ______________________________

Read This First:

- You have three hours to work on the exam.
- The other questions require you to work out answers to the questions; be sure to show your work **clearly**. No credit will be given for illegible work, even if the answers are correct. The number of points each part is worth is indicated in parentheses after the question.
- Work all answers out to two decimal places, when appropriate.
- If a problem is asking for a probability, provide the answer to two decimal places. Do not leave answers in "choose" notation, work them out to the end.
- No sharing of calculators or note sheets during the exam.

PART ONE

Part	**Possible**	**Score**
Base	5	5
MC	12	
1	6	
2	12	
3	10	
4	5	
Total	50	

PART TWO

Part	**Possible**	**Score**
Base	11	11
Page 6	13	
Page 7	9	
Page 8	3	
Page 9	5	
Page 10	12	
Page 11	16	
Page 12	6	
Page 13	14*	
Page 14	7	
Page 15	7	
Total	100	

*Three of these points are extra credit.

PART ONE

Multiple Choice Questions: For each question, circle the one correct letter. Each question is worth 3 points.

1. Suppose A and B are such that $P(A) = P(B) = 0.30$. What is the **largest** $P(A \text{ or } B)$ could be?

   (a) 0.00
   (b) 0.30
   (c) 0.60
   (d) 1.00

2. Random variable X has the binomial($n = 5$, $p = 0.2$) distribution. The probability that X equals one is closest to...

   (a) 0.20
   (b) 0.40
   (c) 0.60
   (d) 0.80
   (e) Cannot determine with the given information.

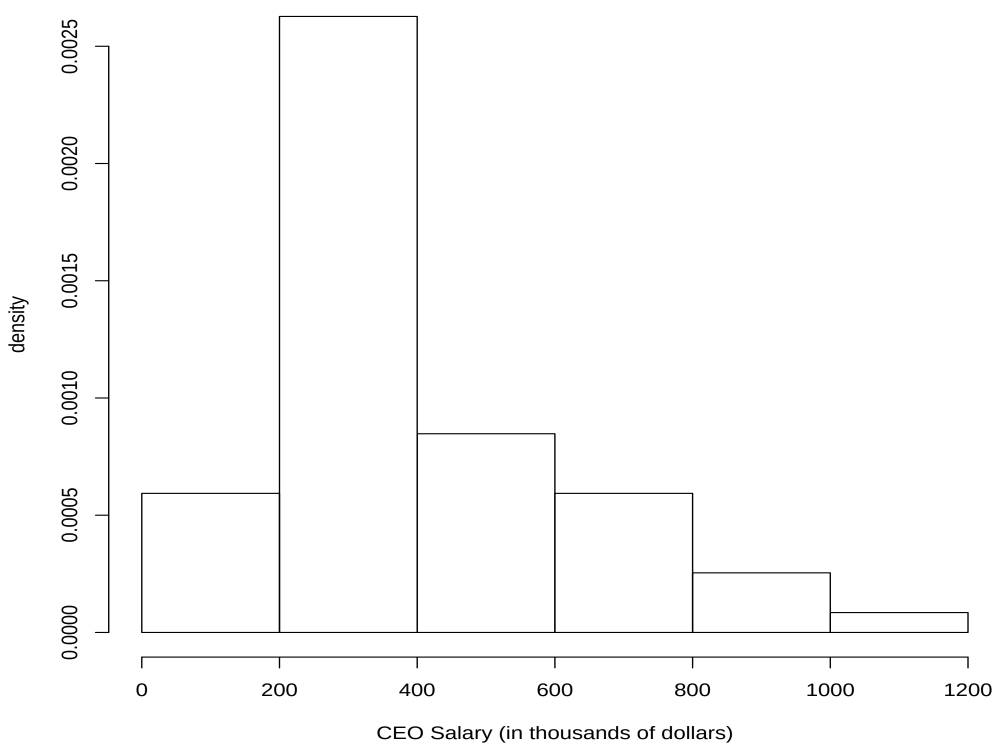

Figure 1: Histogram showing the distribution of salaries in 1993 for CEOs of 59 top small businesses. (Source: *Forbes Magazine*)

3. Figure 1 depicts the distribution of the salary of the CEOs of 59 leading small businesses in 1993. The median salary in this group of 59 salaries is closest to...

   (a) \$50,000
   (b) \$100,000
   (c) \$300,000
   (d) \$600,000

4. Again, refer to Figure 1. The sample mean is larger than the sample median for this set of data. This should not be surprising since...
   (a) the histogram is positively skewed.
   (b) the sample size is fairly large.
   (c) the range of the data is large.
   (d) CEOs are usually paid more than the other employees in their company.

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1. A particular system on the space shuttle consists of two components, named A and B. The probability that component A fails during launch is 0.001, and the probability that component B fails during launch is 0.01. The probability that both fail during launch is 0.0005. Both components need to function correctly for the system to work. What is the probability the system works? (6)

2. The length of the life (measured in days) of a certain brand of light bulb is modeled well by the exponential($\lambda = 0.01$) distribution.
   (a) A light bulb of this brand is chosen at random. What is the probability the length of its life is between 75 and 100 days? (5)
   (b) The light fixture in my living room requires four of these light bulbs. If I install four new ones at the same time, what is the probability that at least one will still be working after 50 days of use? Assume that the bulbs are independent of each other. (4)

(c) Suppose that a light bulb of this brand has been in use for 100 days, and it has yet to fail. What is the probability it lasts at least for 25 **additional** days? (3)

I provide a solution here. If you think it is correct, then write "This solution is correct." If you believe this solution is incorrect, then explain the mistake that is made.

Let X = length of life for one light bulb. Then,

$$\begin{aligned} P(\text{lasts at least 25 more days} \mid \text{has lasted 100 days}) &= P(X \geq 125 \mid X \geq 100) \\ &= \frac{P(X \geq 125 \text{ and } X \geq 100)}{P(X \geq 100)} \\ &= \frac{P(X \geq 125)}{P(X \geq 100)} \\ &= \frac{\exp(-(0.01)(125))}{\exp(-(0.01)(100))} \approx 0.779 \end{aligned}$$

3. The company has two production facilities. The one in Pittsburgh can produce 120 Model A widgets and 50 Model B widgets in a day. The facility in Philadelphia can produce 100 Model A and 60 Model B widgets in a day. Suppose all 330 widgets produced by both factories in one day are brought together.

(a) If two widgets are chosen (without replacement) at random from the group of 330 widgets, what is the probability there is exactly one of Model A and exactly one of Model B? (5)

(b) One widget is chosen at random from the group of 330, and it is of Model A. What is the probability it was produced in the Pittsburgh factory? (5)

4. The probability density function (pdf) of random variable X is

$$f(x) = \begin{cases} 0, & x < 0 \\ 0.2, & 0 \leq x \leq 1 \\ k, & 1 < x \leq 3 \\ 0, & x > 3 \end{cases}$$

What is the probability that X is between 0.5 and 1.5? (Your answer should not include k: figure out what k is.) (5)

PART TWO

1. Which of the following statements describes why the central limit theorem is so important in the application of statistical methods?

   (a) Provided that n is sufficiently large, it allows one to approximate the distribution of the sample mean ($\overline{X}$) without needing to know the distribution of $X_1, X_2, \ldots, X_n$.

   (b) It allows for the construction of probability plots.

   (c) It shows that if X is a normal random variable with mean μ and variance σ^2, then $(X - \mu)/\sigma$ has the standard normal distribution.

   (d) It allows one to approximate the probability that X is less than a, for any value a, without knowing the distribution of X.

Questions 2 and 3 refer to the following situation: The Brand A bolts currently in use break, on average, when the weight they hold reaches 150 pounds. The Brand B bolt is being considered. Switching the brand of bolt will come at significant expense, but if there is evidence that Brand B bolts can withstand, on average, more than 150 pounds, the switch will be made.

2. What would be the **type I Error** in this case?

   (a) 0.05

   (b) Deciding to stay with Brand A when Brand B has higher average breaking point than Brand A.

   (c) Deciding to stay with Brand A when Brand B has lower average breaking point than Brand A.

   (d) Deciding to switch to Brand B when Brand B does not have higher average breaking point than Brand A.

   (e) Using a sample size which is too large.

3. A sample of 45 Brand B bolts are chosen and tested. The average breaking point for the sample of bolts is 156.8 pounds, and the sample standard deviation is 20.9. Is there strong evidence that Brand B has higher average breaking point than Brand A?

   State the P-value, or choose an acceptable probability of type I error (i.e. choose α) and come to a conclusion. Either way, write out your conclusions in one sentence. Should the company switch brands? (7)

Questions 4, and 5 refer to the following situation: The production department predicted the customer demand for each of four colors of shirt. A sample of 100 sales are tracked, and the colors of the shirts sold are recorded. The following table shows the predicted proportions and color counts in the sample.

Color	**Prediction**	**Sample Count**
Red	0.30	25
Blue	0.25	28
Black	0.25	28
Yellow	0.20	19

4. First, suppose the goal is to see if there is strong evidence that the actual color distribution deviated from the predicted color distribution. In this situation we should...

   (a) calculate the mean squared error.

   (b) use a goodness-of-fit hypothesis test.

   (c) use a two-sample t test.

   (d) use an analysis of variance (ANOVA) procedure.

5. Next, suppose the goal is to estimate the proportion of all of the shirt sales that were red (ignoring the prediction). We know that the sample proportion of red shirts sold, denoted $\widehat{p}$, is approximately normal with mean p and (approximate) variance $\widehat{p}\,(1-\widehat{p})/n$. Thus,

$$P\left(-z_{\alpha/2} \leq \frac{\widehat{p}-p}{\sqrt{\widehat{p}\,(1-\widehat{p})/n}} \leq z_{\alpha/2}\right) = 1-\alpha.$$

   This can be rearranged to show that

$$P\left(\widehat{p} - z_{\alpha/2}\sqrt{\frac{\widehat{p}\,(1-\widehat{p})}{n}} \;\leq\; p \;\leq\; \widehat{p} + z_{\alpha/2}\sqrt{\frac{\widehat{p}\,(1-\widehat{p})}{n}}\right) = 1-\alpha.$$

   Use this to form a 95% confidence interval for the unknown p, the proportion of all the shirt sales that were red. (6)

Questions 6 and 7 refer to the following situation: From Devore, Exercise 12.46:

> The article "The Incorporation of Uranium and Silver by Hydrothermally Synthesized Galena" (*Econ. Geology,* 1964: 1003-1024) reports on the determination of silver content of galena crystals grown in a closed hydrothermal system over a range of temperature. With $x =$crystallization temperature in degC and $y = Ag_2S$ in mol%.

Figure 2 shows the scatter plot of Ag_2S versus crystallization temperature. Note that there are $n = 12$ data pairs. (I removed one potential "outlier" from the data set.)

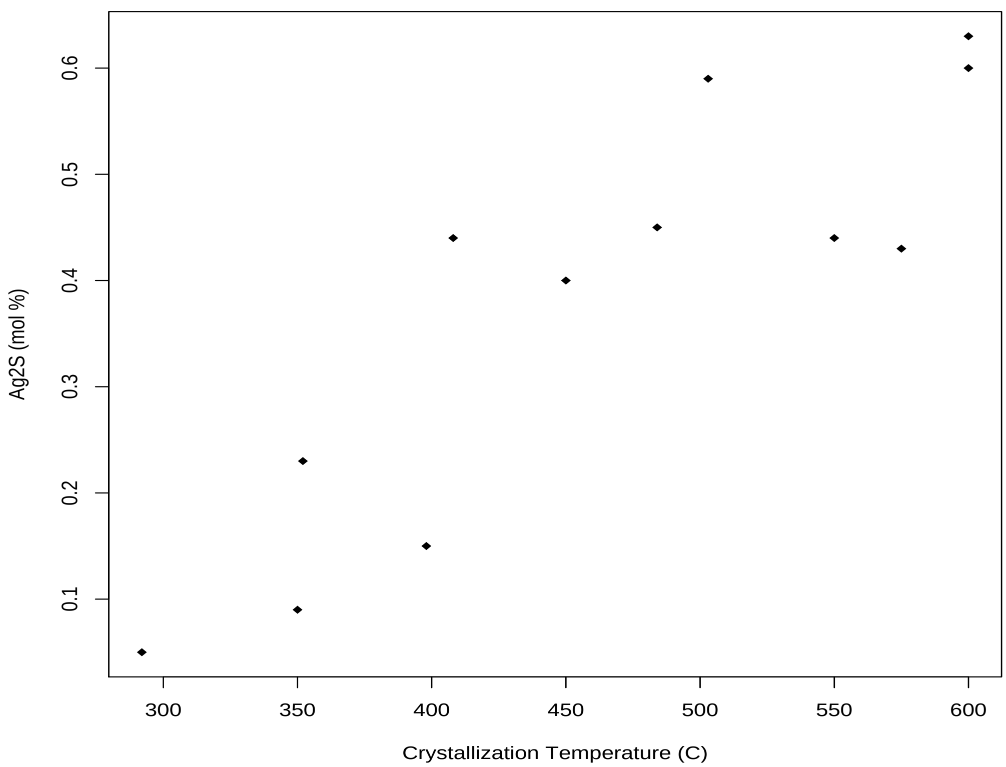

Figure 2: Scatter plot of Ag_2S versus crystallization temperature for Questions 6 and 7.

A linear model is fit to the data, i.e. the parameters β_0 and β_1 are estimated in the following equation.

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

Below is a portion of the computer output (using the statistical analysis package "R").

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.3973155  0.1367342  -2.906 0.015678
(Slope)      0.0016663  0.0002883   5.780 0.000178
```

Finally, Figure 3 shows the plot of residuals versus fitted values for this case.

6. Does the linear model seem appropriate in this case? Why or why not?

   (a) Yes, since there is a linear pattern in both the scatter plot and the plot of residuals versus fitted values.

   (b) No, since there is not a linear pattern in the plot of residuals versus fitted values.

   (c) No, the estimate of β_1 is too small.

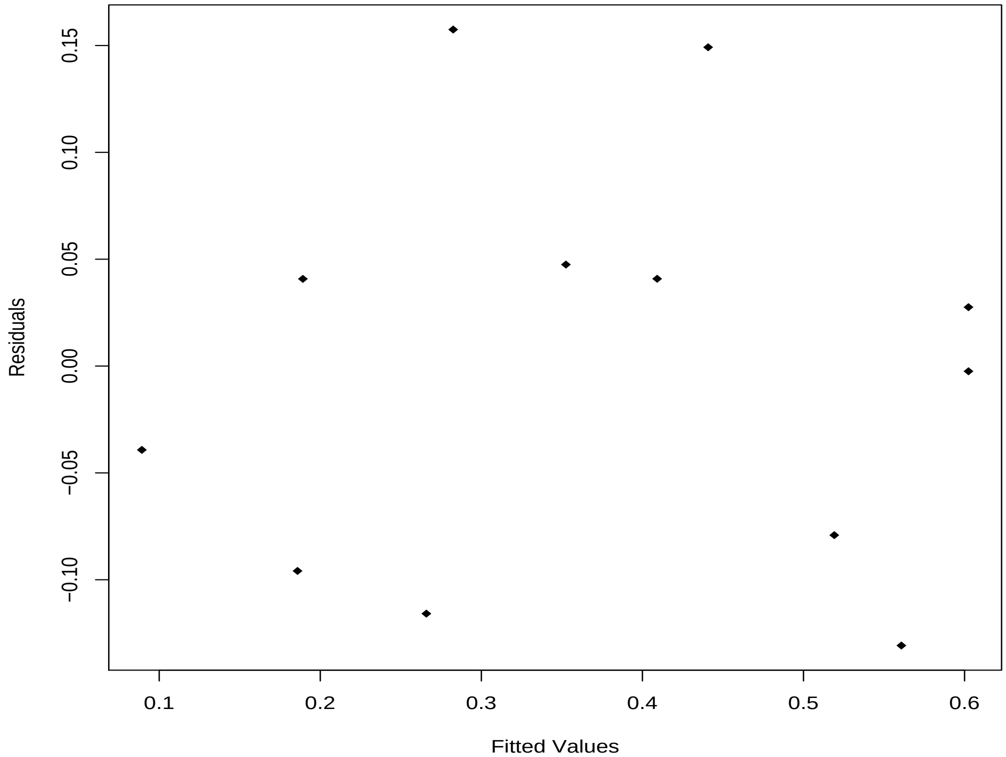

Figure 3: Plot of residuals versus fitted values for Questions 6 and 7.

(d) Yes, since there is a linear pattern in the scatter plot and there is no noticeable pattern in the plot of residuals versus fitted values.

7. Form a 95% confidence interval for β_1, the slope of the regression line. Assume that the model errors ϵ_i are normally distributed with mean zero and variance σ^2. (5)

Questions 8, 9, and 10 refer to the following situation: Suppose that random variable X has the binomial$(n = 10, p)$ distribution, with p unknown. The objective is to determine if there is strong evidence to claim p is larger than 0.4. So, we will test $H_0: p = 0.4$ versus $H_a: p > 0.4$.

8. What would the P-value for this test be if we were to observe a value of 8 for X? (6)

9. Suppose you decide that you will reject the null hypothesis only if you observe a value of either 9 or 10 for X. What is the probability of type I error? (3)

10. Again, suppose you will reject the null hypothesis only if you observe a value of either 9 or 10 for X. What is the probability of type II error if the true value of p is 0.5? (3)

Questions 11,12, 13, and 14 refer to the following situation: Read the following dialogue from a meeting between an engineer and her boss.

> **Boss:** What progress has been made in reducing the number of defective widgets produced on line ten? For the longest time, 10% of the widgets produced on that line have been bad.
>
> **Engineer:** I have implemented some improvements, and I am confident that the proportion of defectives has been reduced below 10%.
>
> **Boss:** That would be impressive. If you can provide me with strong evidence that the rate of defectives has been brought below 10%, I would give you a significant raise.
>
> **Engineer:** I will test a sample of widgets and see what proportion are defective.
>
> **Boss:** But I want strong proof. If there really is no improvement, I want the probability I give you a raise to be at most 5%.
>
> **Engineer:** I understand, you want the probability of type I error to be at most 5%.
>
> **Boss:** Yes, and keep in mind that your tests will cost money. In order to test a widget, it must be dismantled and inspected. It can no longer be used after that. Don't test a large sample, we can't afford it.
>
> **Engineer:** I know, but I want a reasonable chance of getting the raise. You want to avoid a type I error, but I want to avoid a type II error, which in this case would occur if I really did improve the rate of defectives, but you do not give me the raise.
>
> *The engineer leaves and prepares a graph, shown as Figure 4.*
>
> **Engineer:** Here I have a plot showing the probability of type II error as a function of sample size (n) and the true proportion of defectives (p).

11. Again, let p denote the true proportion of defectives. What hypothesis test is being discussed here?
    (a) $H_0: p = 0.10$ versus $H_a: p < 0.10$.
    (b) $H_0: p = 0.10$ versus $H_a: p > 0.10$.
    (c) $H_0: p = 0.05$ versus $H_a: p < 0.05$.
    (d) $H_0: p = 0.05$ versus $H_a: p > 0.05$.
    (e) None of the above.

12. There are three curves on Figure 4, one represents the case where $p = 0.03$, one where $p = 0.05$, and one where $p = 0.09$. Complete the labels on the plot. (5)

13. Suppose the boss agrees that the probability of type II error should be at most 0.20 if, in fact, $p = 0.05$. How large of a sample should the engineer test? (Approximate the answer using the graph.) (5)

14. The engineer believes that her boss is being unfair. She says, "Using that sample size, the probability I get the raise is around 10% if, in fact, I reduced the proportion of defectives to 9%." The boss replies, "I know that if I choose __________ large enough, we can get the probability of type II error when $p = 0.09$ to be small, but not only would that cost us a lot of money, but also in __________ terms $p = 0.09$ is not a large enough improvement for you to merit a raise." Choose the two terms which correctly completes his reply.
    (a) α / statistical
    (b) $\beta(0.09)$ / objective
    (c) the sample size / practical
    (d) the size of your raise / theoretical

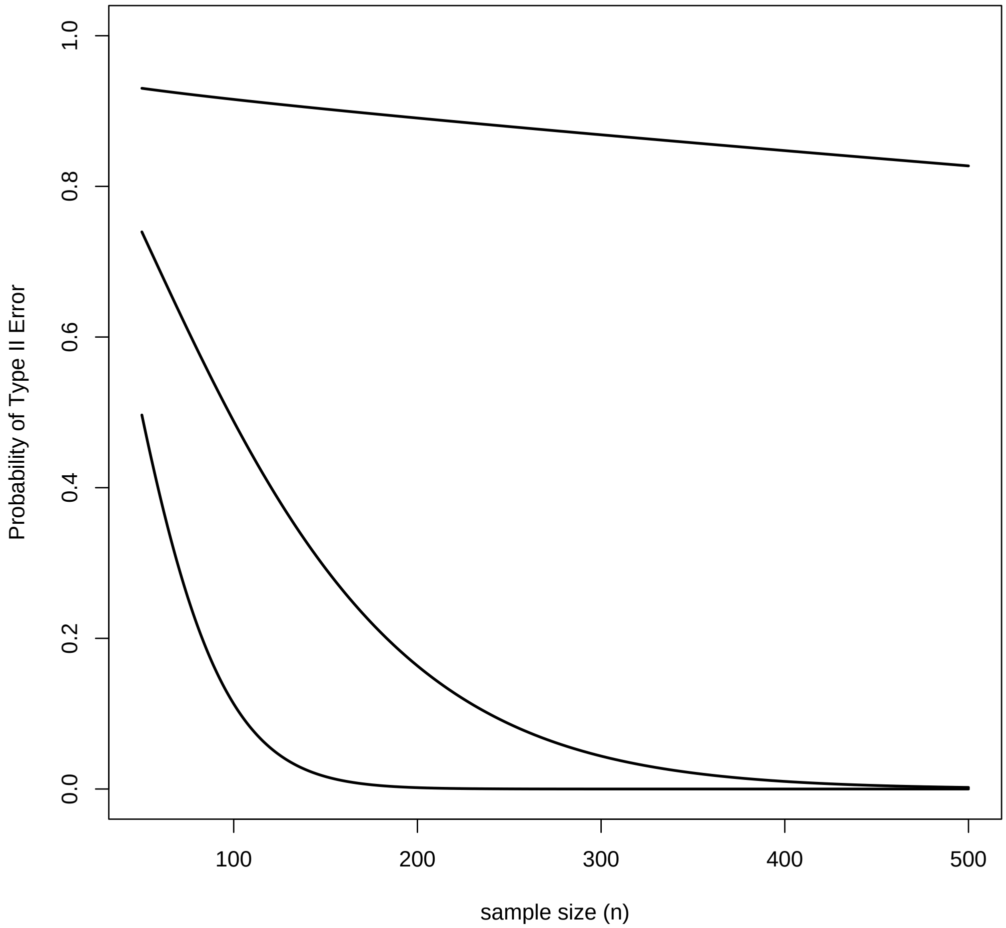

Figure 4: Plot of probability of type II error, as a function of sample size (n) and proportion of defective (p). Used in Questions 12, 13, and 14.

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15. Suppose that $X_1, X_2, \ldots, X_n$ are a sequence of independent random variables, each with the same distribution. What does the *Law of Averages* state?

    (a) As the sample size gets larger, the sample total T_0 will get closer and closer to the expected value of the population total.

    (b) As the sample size gets larger, the variance of the sample mean gets closer and closer to one.

    (c) As the sample size (n) gets larger, the sample mean $\overline{X}$ will get closer and closer to the population mean.

    (d) As the sample size gets larger, the sample standard deviation (s) will get smaller.

16. Random variables X_1 and X_2 are independent, both are normally distributed, and each has mean 12.0 and variance 4.0. The distribution of

$$\frac{X_1 + X_2}{2}$$

    is which of the following?

    (a) Normal with mean 24.0 and variance 2.0.

    (b) Normal with mean 24.0 and variance 4.0.

    (c) Normal with mean 12.0 and variance 2.0.

    (d) Standard normal

    (e) Cannot determine from the given information.

Questions 17, 18, 19 and 20 refer to the following situation: You desire to weigh an object, and have two scales to choose from. The scales are not exact, i.e. they do not return the exact weight of the object placed upon it. If the object has weight θ, then the weight returned by Scale One is a normal random variable with mean θ and variance 1 mg^2. The weight given by Scale Two is a normal random variable with mean θ and variance 4 mg^2.

Label the weight given by Scale One as X_1, and the weight given by Scale Two as X_2. Estimate the weight of the object by

$$\widehat{\theta} = aX_1 + (1-a)X_2$$

where a is a number between 0 and 1. Assume X_1 and X_2 are independent.

17. Show that, for any value a, $\widehat{\theta}$ is an unbiased estimator of θ. (4)

18. What is the variance of $\widehat{\theta}$? (It will depend on a, simplify the expression as much as possible.) (4)

19. Figure 5 shows the mean squared error (MSE) of $\widehat{\theta}$ as a function of a. What would be the best choice for a? (Approximate it using the plot.)(3)

20. **EXTRA CREDIT:** Determine the best choice for a exactly. (You need to show the derivation.) (3)

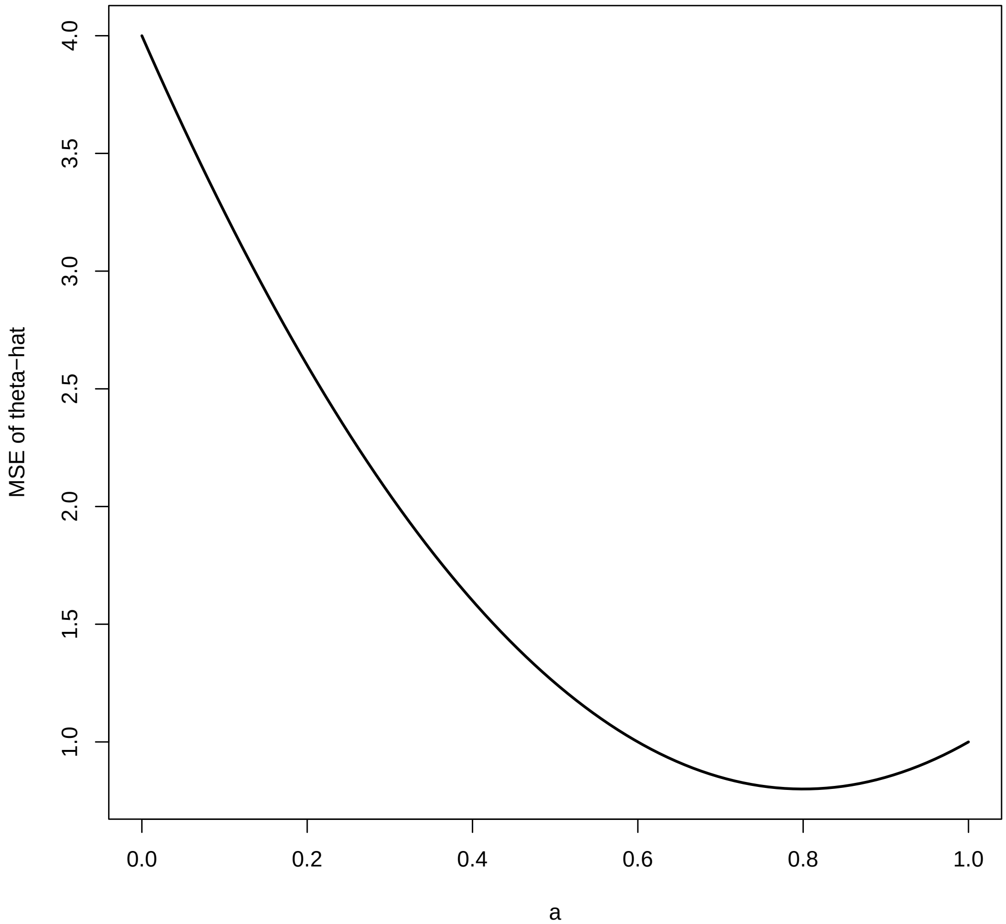

Figure 5: Plot of MSE of $\widehat{\theta}$ as a function of a. Used in Question 19.

Questions 21 and 22 refer to the following situation: Exercise 7.13 in Devore:

> The article "Extravisual Damage Detection? Defining the Standard Normal Tree" (*Photogrammetric Engr. and Remote Sensing,* 1981: 515-522) discusses the use of color infrared photography in identification of normal trees in Douglas fir stands. Among data report were summary statistics for green-filter analytic optical densitometric measurements on samples of both healthy and diseased trees. For a sample of 69 healthy trees, the sample mean dye-layer density was 1.028, and the sample standard deviation was 0.163.

21. Calculate a 99% confidence interval for the true average dye-layer density for all such trees. (7)

22. Suppose the investigators had made a rough guess of 0.16 for the value of σ before collecting the data. What sample size would be necessary to obtain an interval width of 0.05 for a confidence level of 99%? (4)

23. In a wide range of situations, the width of the confidence interval is equal to $k/\sqrt{n}$ where k is a constant whose value depends on the situation. For example, in the Question 22 above the width of the confidence interval is

$$2\,(z_{.005})\sqrt{\frac{0.16^2}{n}},$$

so that would mean $k = 2(z_{.005})(0.16)$.

Now complete this statement: If the width of the confidence interval can be written $k/\sqrt{n}$ (which it can be in many situations), then if I want to cut the width of the confidence interval in half, I need to multiply my sample size n by ___________. (3)