Assignment 1

36-462, Spring 2008

Due 1 February 2008

When a problem asks you to do a simulation, include your code in your answer.

1. BIFURCATIONS IN THE LOGISTIC MAP

- (a) Find the value of r at which the fixed point bifurcates into a cycle with period 2; call this r_2 .
- (b) Verify that the fixed point is stable when $r < r_2$ and unstable when $r > r_2$.
- (c) Verify, using the stability criterion in Flake, that the 2-cycle is stable for values of *r* just above *r*₂.
- (d) Find the r at which the 2-cycle becomes unstable, and show that a 4-cycle appears there.

Hints: (i) look at the material in lecture 2 on how the fixed point at zero becomes unstable; (ii) look up how to solve a quartic (not quadratic) equation.

2. LYAPUNOV EXPONENTS IN THE LOGISTIC MAP For a one-dimensional map Φ , the Lyapunov exponent is the limiting time-average of the log derivative:

$$\lambda = \lim_{T \to \infty} \sum_{t=1}^{T} \log |\Phi'(x_t)|$$

- (a) Use this fact to write a program to calculate the Lyapunov exponent of the logistic map as a function of *r*.
- (b) Plot λ as a function of r.
- (c) Explain how this plot is related to the bifurcation diagram. *Hint:* make a new plot of λ , with *r* confined to the chaotic region.
- 3. ROTATIONS This is a one-dimensional map for which the state θ_t is an angle on the circle, measured in degrees, i.e. $0 \le \theta_t < 360$. The circle is rotated by a constant angle α each time-step.

$$\theta_{t+1} = \theta_t + \alpha \bmod 360$$

- (a) Show (by algebra, not simulation) that if α is rational then every point is a periodic point, but if α is irrational than there are no periodic points.
- (b) Show, by simulation, that this map is ergodic when α is irrational, with the invariant distribution being the uniform distribution.¹
- (c) Show, again by simulation, that this map is not mixing.
- 4. THE RÖSSLER ATTRACTOR Install the tseriesChaos package from CRAN. (Be sure to install any required packages as well.) Run the command

(Or type example (sim.cont) to see this again.) This generates a time series from the dynamical system known as the Rössler equations. You could look them up, but then you wouldn't learn much from this exercise.

- (a) Reconstruct the attractor using the method of false nearest neighbors. Explain how you determined the time lag and the embedding dimension. (It is no more than 9.) Include a plot of the reconstructed attractor, if physically possible.
- (b) The embedding dimension k can also be thought of as a setting to be chosen by cross-validation. Which embedding dimension, in the range of 1 to 9, gives the smallest cross-validated prediction error for 1-nearest-neighbor prediction? (Warning: Allow your computer a lot of time to think about this!) If this answer is not the same as in the previous part, why do you think they differ?
- 5. MIXING ON THE HÉNON ATTRACTOR For simplicity, consider only the distribution of the *x* coordinate, so that you can make one-dimensional histograms.
 - (a) Find, by simulation, the long-run distribution of points visited from a single trajectory of the Hénon map when a = 1.29, b = 0.3. Does the answer depend on the initial condition?
 - (b) Experiment with different ensembles of initial conditions; how much does the distribution of points at t = 100 vary?
 - (c) Show by simulation that the time-average of x_t converges to the same value, independent of the initial condition.
 - (d) Does $\sqrt{T} \frac{1}{T} \sum_{t=1}^{T} X_t$ go towards a Gaussian distribution for large *T*?

¹This was first proved formally, in rather different terms, by Nicholas Oresme in the 1300s.