Chaos, Complexity, and Inference (36-462)

Lecture 2

Cosma Shalizi

17 January 2008

Stability of fixed points

A **fixed point** or **equilibrium** x^* is a solution to

$$x = f(x)$$

If $x_1 = x^*$, then $x_t = x^*$ forever
What if x_1 is not a fixed point? What if someone knocks the trajectory off the fixed point ever so slightly?
Taylor expansion around x^*

$$\begin{aligned} x_{t+1} &= f(x^*) + (x_t - x^*)f'(x^*) + \text{remainder} \\ x_{t+1} &= x^* + (x_t - x^*)f'(x^*) + \text{remainder} \\ x_{t+1} - x^* &= (x_t - x^*)f'(x^*) + \text{remainder} \end{aligned}$$

back to exponential growth or decay, supposing that the remainder term is in fact small

$\lvert f'(x^*)\rvert$	fate of small perturbations	label
$= 0$	super-exponential decay	super-stable
< 1	exponential decay	stable
$= 1$	set by remainder in Taylor expansion	neutral
> 1	exponential growth	unstable

For a cycle $x_1, x_2, \ldots x_p$, evaluate

$$\left|\prod_{i=1}^{p} f'(x_i)\right|$$

— similar but more tedious calculus

All initial conditions sufficiently close to a stable fixed point approach that fixed point

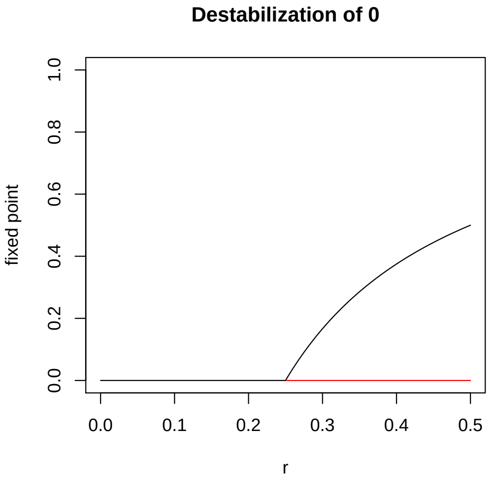
All initial conditions sufficiently close to an *unstable* fixed point move away from it

Ditto for limit cycles

Bifurcation What happens to the stability of a fixed point as the control parameters are changed?
Example: stability of $x = 0$ in the logistic map
$f'(0) = 4r$
Stable if $r < 0.25$, unstable if $r > 0.25$, neutral if $r = 0.25$
Suppose $r = 0.25 + h$, $h > 0$
Solve:

$$\begin{aligned} x &= (1 + 4h)x(1 - x) \\ x &= x + 4hx - x^2 - 4hx^2 \\ 0 &= x(4h - x - 4hx) \\ \frac{4h}{1 + 4h} &= x \end{aligned}$$

When h is small, $x^* \approx 4h - 16h^2$

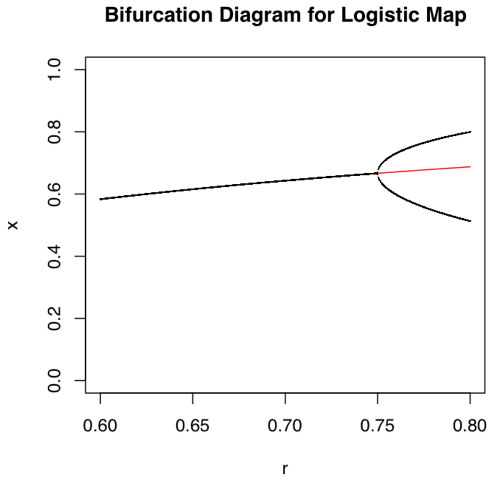

Destabilization of 0
fixed point
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5
r

This is a simple example of a **bifurcation**, a point in parameter space (not state space) where the stability of solutions changes qualitatively

Bifurcation diagram = plot of stable solutions vs. control parameters

Easiest way to make one: fix r, take a random x_1, calculate x_T for T large, then plot $x_{T+1}, x_{T+2}, \ldots x_{T+t}$ ("throw away transients"); repeat for another value of r

```
plot.logistic.map.bifurcations <- function(from=0,to=1,n=201,
                                           plotted.points=1000
                                           transients=10000) {
  r.values = seq(from=from,to=to,length.out=n)
  total.time = transients+plotted.points
  plot(NULL,NULL,xlab="r",ylab="x",xlim=c(from,to),ylim=c(0,1)
       main="Bifurcation Diagram for Logistic Map")
  for (r in r.values) {
    x = logistic.map.ts(total.time,r)
    x = x[(transients+1):total.time]
    points(rep(r,times=plotted.points),x,cex=0.01)
  }
}

> plot.logistic.map.bifurcations(from=0.6,to=0.8,n=501)
> curve(4*(x-0.25)/(1+4*(x-0.25)),from=0.75,to=0.8,add=TRUE,co
```


Bifurcation Diagram for Logistic Map
x
0.0
0.2
0.4
0.6
0.8
1.0
0.60
0.65
0.70
0.75
0.80
r

What happens at $r = 0.75$?
Period 2 means points x_a, x_b which solve

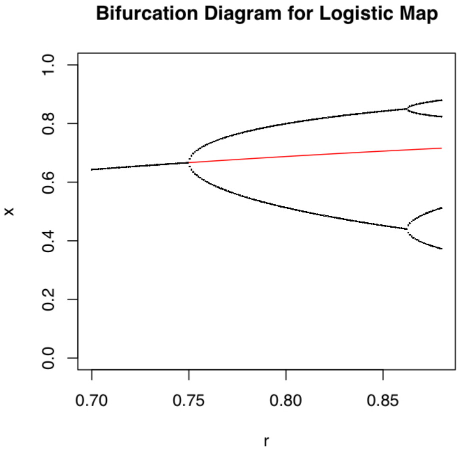
$$x = f(f(x)) \tag{1}$$

and where

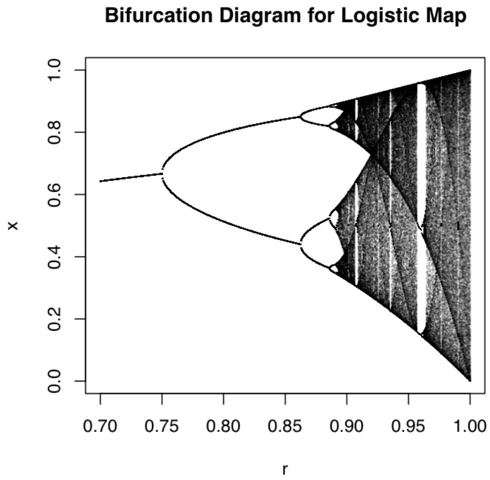
$$\begin{aligned} x_a &= f(x_b) \\ x_b &= f(x_a) \end{aligned} \tag{2}$$

If $x^* = f(x)$ then $f(f(x^*)) = f(x^*) = x^*$ so if there are fixed points then there are solutions to (1)
But maybe there are none which *also* solve (2)
At $r = 0.75$, $x = f(f(x))$ goes from having only 2 distinct solutions to have 4
and (2) gets solutions
and both solutions of $x = f(x)$ become unstable
and the periodic solution is stable
You can check this using `polyroot` and the stability rules

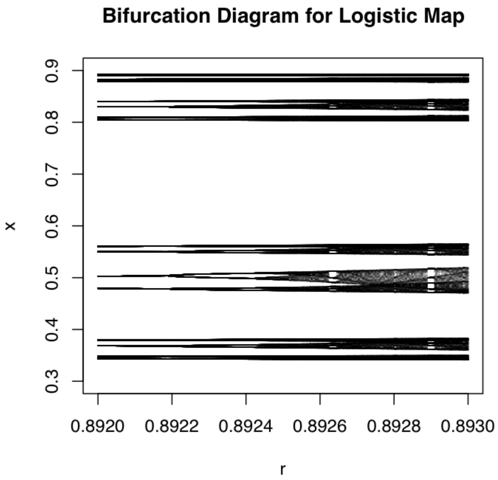
Next bifurcation: 2-cycle destabilizes, stable 4-cycle appears



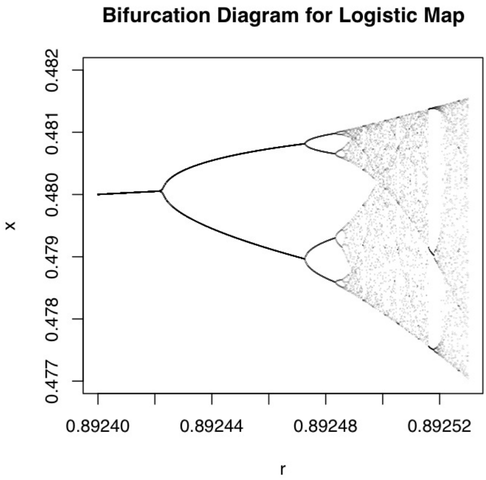
Overall picture



Each branch keeps splitting in 2



Each split looks like the first, scaled down



This keeps on happening, so more and more bifurcations pile up
In the limit distance between successive bifurcations shrinking by factor of 4.69...
Infinitely many bifurcations between $r = 0.25$ and $r \approx 0.89248$ — the **period-doubling accumulation point**
What happens when there are infinitely many periodic orbits, with infinite periods, but they are all unstable?

Chaos

A more formal definition of chaos (due to Devaney):

1. There are periodic orbits arbitrarily close to any given point.
   which means there are infinitely many periodic points
   which means there are infinitely many periodic cycles
2. The map is *transitive*, i.e. there is some orbit connecting any two regions

Together these imply

sensitive dependence on initial conditions There is a sensitivity scale δ such that any two orbits will *eventually* be at least δ apart, no matter how close they started.

Sketch: periodic points stay on their cycle; but arbitrarily close to any periodic point is a wandering point which eventually gets arbitrarily close to a *different* cycle.
Take-home: chaos always has an infinity of periodic structures embedded in it, but they're all unstable

Mechanically, chaos requires “stretching and folding”
Stretch: locally, separate near-by points in the state space
Fold: then stuff everything back into the state space
At $r = 1$, each *half* of the state space is mapped on to the whole

Fun with stretching and folding: Arnold Cat Map

Our first two-dimensional map!

$$(x_{t+1}, y_{t+1}) = (x_t + y_t, x_t + 2y_t) \text{ mod } 1$$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \text{ mod } 1$$

This is ergodic and even mixing, but also exactly reversible

And: embedded periodic points (rational numbers)

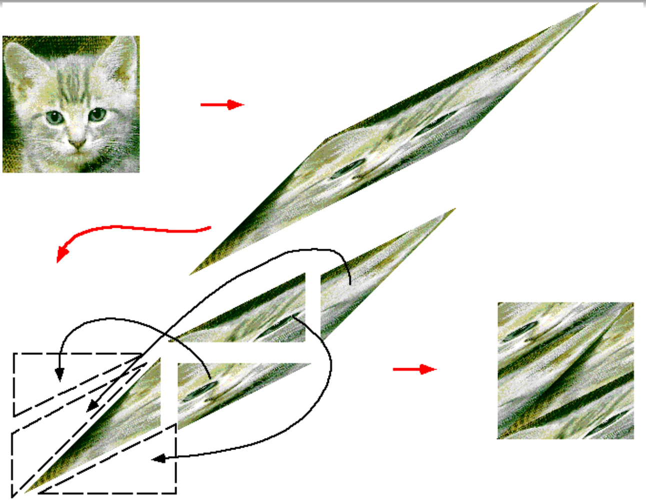
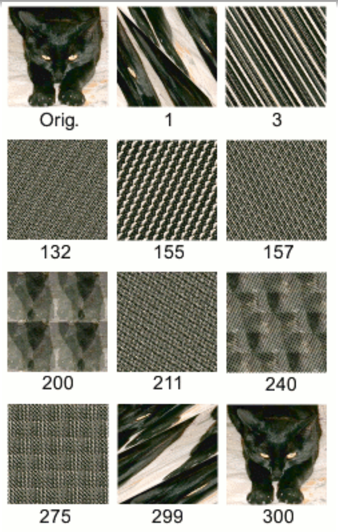
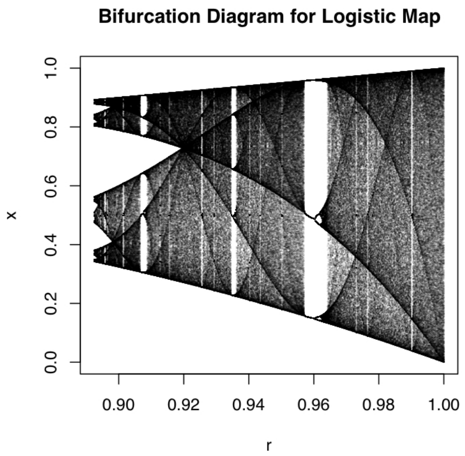


Image from Leon Poon http://www-chaos.umd.edu/images/catmap.gif



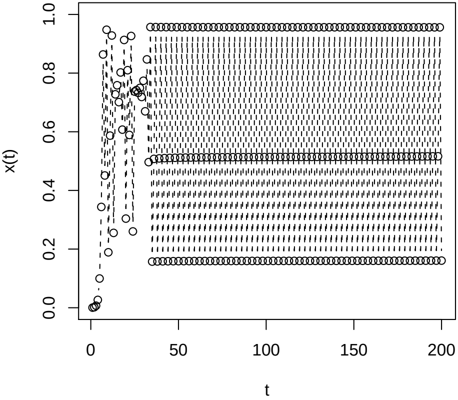
from Wikipedia, s.v. “Arnold’s cat map”

Periodic windows break up the chaotic region; each has its own period-doubling cascade

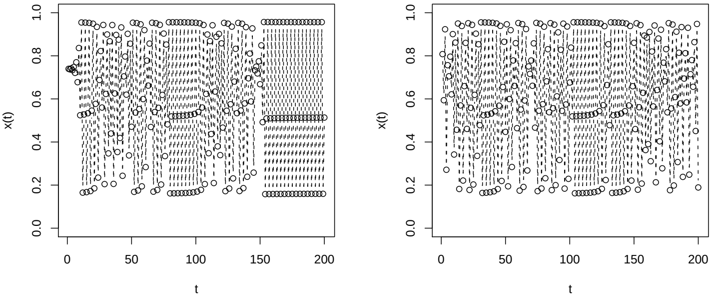


Intermittency

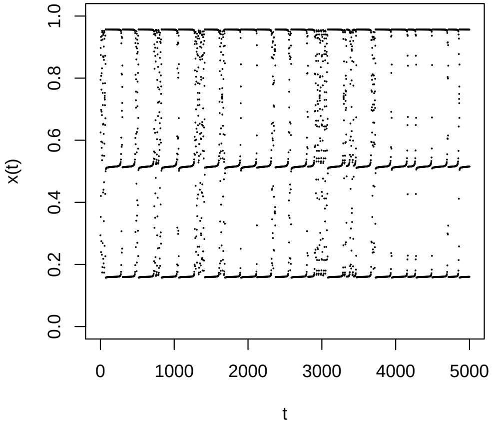
Take $r = 0.9571$



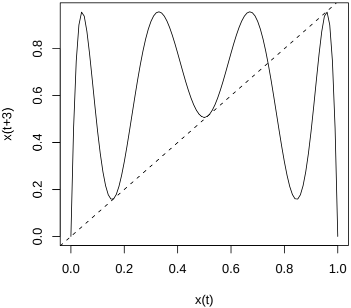
Just a stable 3-cycle? Try some more initial conditions



take a longer view

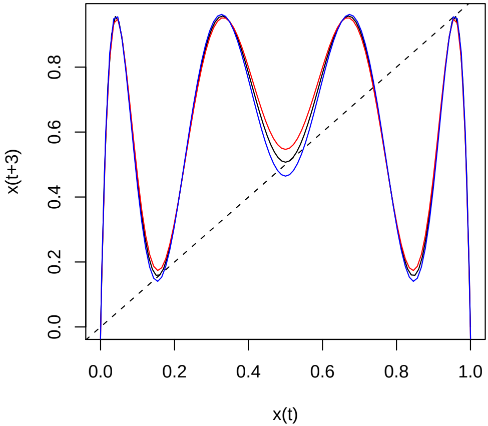


Trajectories switch between staying at a 3-cycle and looking properly chaotic
This is **intermittency** or **intermittent chaos**
Look at plot of $f(f(f(x))) \equiv f^{(3)}(x)$ to understand

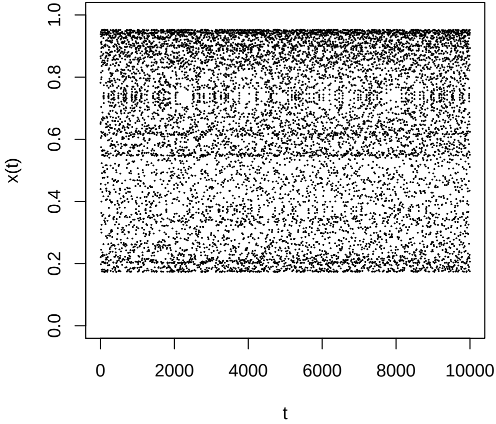


5 fixed points — 3-cycle + 2 unstable — periodic points are *tangent* to diagonal so all neutrally stable

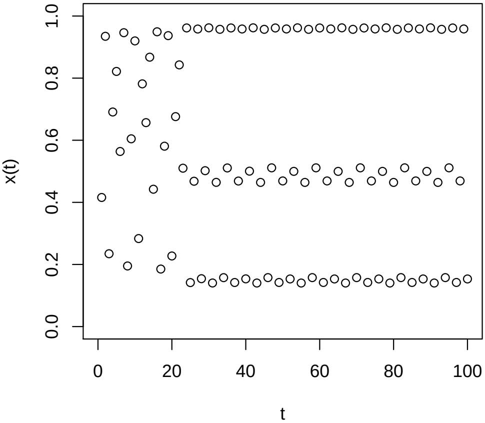
Now add in what happens with r at bit bigger (red) and a bit smaller (blue)



r a bit smaller: two unstable solutions points (at 0 and around $3/4$)

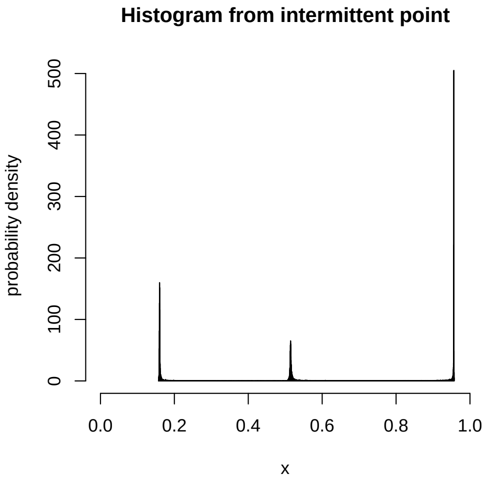


r a bit larger: eight solutions (zero, unstable fixed point, stable 6-cycle)

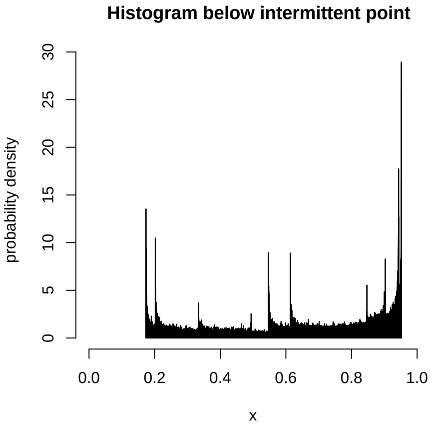


Notice that because the $f^{(3)}$ curve is *tangent* to the diagonal that the derivative $= 1$ at every periodic point, so it is neutrally stable
When the orbit comes close to one of the periodic points, it stays there for a long time, the orbit is *almost* stable

Distribution: spikes around the points that want to be a 3-cycle



Distribution just below intermittent point



Note: histogram stays in range of $f^{(3)}$
Note: difference in vertical scale

Generically, the invariant distributions of chaos are very irregular and spiky
We will come back to this next lecture

At a more practical level: chaos means determinism, sensitivity, *and* ergodicity

Ergodicity

More precise definition than last time: for almost any initial condition x_1 and any reasonable function h

$$\frac{1}{n}\sum_{t=1}^{n} h(x_t) \xrightarrow[n\to\infty]{} \int h(x)\rho(x)dx$$

where $\rho(x)$ is the invariant density
Left-hand side is a **time average**
Right-hand side is an **expectation** or **(state) space average**
Ergodicity means “time averages converge on expectations”

More on the evolution of ensembles

Remember the transformation formula for densities: if X has density p, then $Y = f(X)$ has density q with

$$q(y) = p(f^{-1}(y))\left|\frac{\partial f}{\partial x}\right|^{-1}$$

taking the derivative at $f^{-1}(y)$ as well
meaning: density at new point is density at the point w came from, tiems the size of the region which goes there
still works for maps but now for $f^{-1}(x_{t+1})$ can have multiple values; add up terms for each one

for logistic map with $r = 1$

$$p_{t+1}(x) = \frac{p_t(0.5 - 0.5\sqrt{1-x})}{4\sqrt{1-x}} + \frac{p_t(0.5 + 0.5\sqrt{1-x})}{4\sqrt{1-x}}$$

Can be used to evolve densities exactly, rather than by simulation
Exercise: Show that this really does leave the invariant distribution alone
N.B. the evolution of the ensemble is linear!

Very simple ergodic systems:

- Fixed points (invariant distribution puts all probability on fixed point)
- Periodic cycles (invariant distribution puts equal probability on each point)

At the other end: if $x_1, x_2, \ldots$ are successive IID random samples, then law of large numbers $\equiv$ ergodic property

Very Simple Ergodic Theorem

X_t are random variables with constant mean and variance,
$\mathrm{cov}[X_t, X_{t+\tau}] = \Gamma(\tau)$
IF

$$\frac{\sum_{\tau=0}^{\infty} |\Gamma(\tau)|}{\Gamma(0)} \equiv \tau_{\mathrm{corr}} < \infty$$

THEN

$$\mathrm{var}\left[\frac{1}{T}\sum_{t=1}^{T} X_t\right] \xrightarrow[T\to\infty]{} 0$$

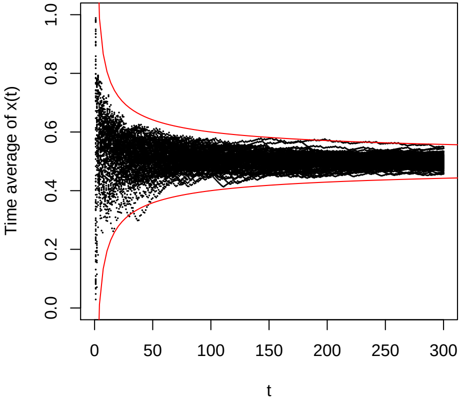
SO with time averages converge stochastically on expectations
($\Leftarrow$ variance $\downarrow 0$ + Chebyshev's inequality)
take $\mathbf{E}[X_t] = 0$ for simplicity

$$\begin{aligned}
\mathbf{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T} X_t\right)^2\right] &= \mathbf{E}\left[\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} X_t X_s\right] = \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{cov}[X_t, X_s] \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma(t-s) \\
&= \frac{2}{T^2}\sum_{t=1}^{T}\sum_{\tau=0}^{t}\Gamma(\tau) \\
&\leq \frac{2}{T^2}\sum_{t=1}^{T}\sum_{\tau=0}^{\infty}|\Gamma(\tau)| \\
&= \frac{2}{T}\sum_{\tau=0}^{\infty}|\Gamma(\tau)| \\
&= \frac{2}{T}\Gamma(0)\tau_{\mathrm{corr}}
\end{aligned}$$

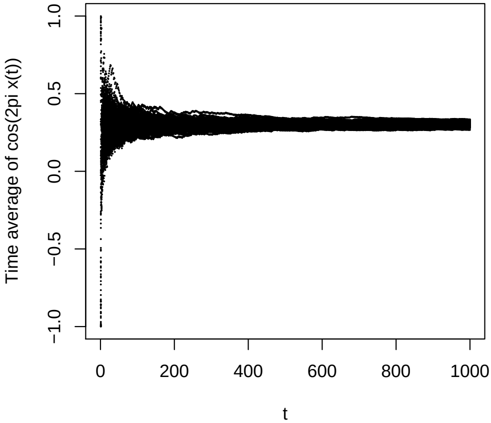
Variance $\propto 1/T$, just like variance of a random sample $\propto 1/N$
but correction factor of $\tau_{\mathrm{corr}} \approx$ time needed for correlation to decay
Notice that this is sufficient, not necessary, for ergodic convergence, because correlations do *not* decay for periodic cycles

```
plot.logistic.map.timeaverages <- function(timelength,num.traj
                                            r,lined=TRUE,cex=1)
  plot(NULL,NULL,xlim=c(0,timelength),ylim=c(0,1),xlab="t",
       ylab="Time average of x(t)")
  i = 0
  while (i < num.traj) {
    i <- i+1
    x <- logistic.map.ts(timelength,r)
    x.avg = cumsum(x)/(1:timelength)
    if (lined==TRUE) {
      lines(1:timelength,x.avg,lty=2)
    }
    points(1:timelength,x.avg,cex=cex)
  }
}
```

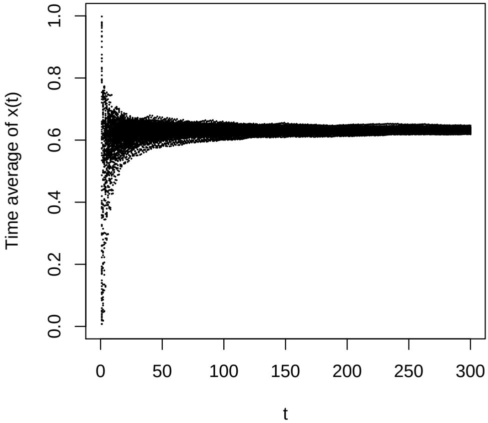
Time-averages of logistic map, $r = 1$, with $1/\sqrt{t}$ lines
Recall $\mathrm{cov}[X_t, X_{t+1}] = 0$, similarly $\Gamma(\tau) = 0$ always



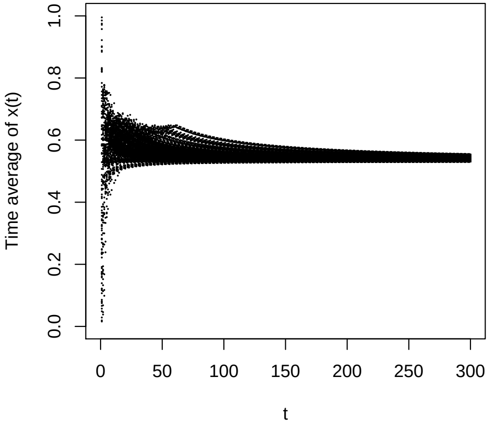
Time average of $\cos 2\pi x_t$, because we can ($r = 1$)



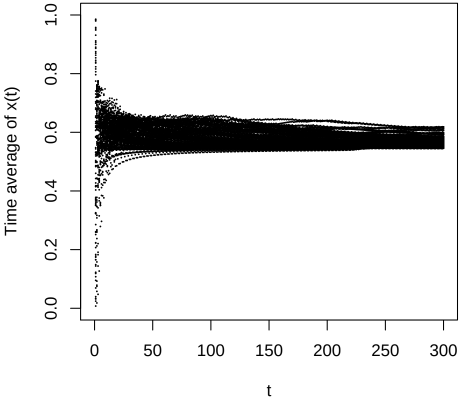
$r = 0.9521$ (below intermittency)



$r = 0.9621$ (above intermittency)

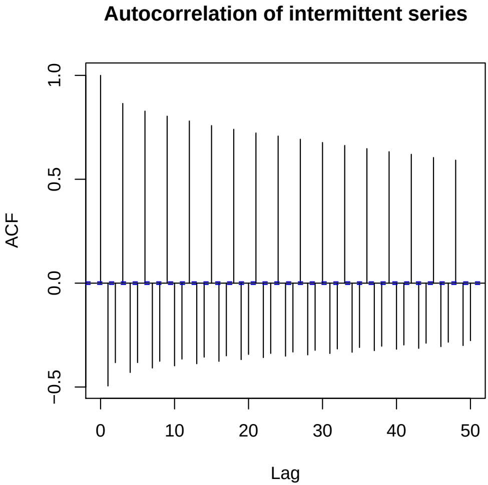


$r = 0.9571$ (intermittency)



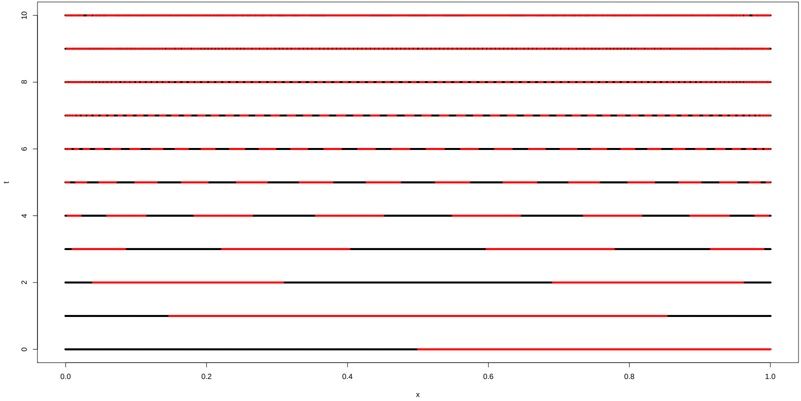
note slower convergence

Intermittency means correlations, but they do decay



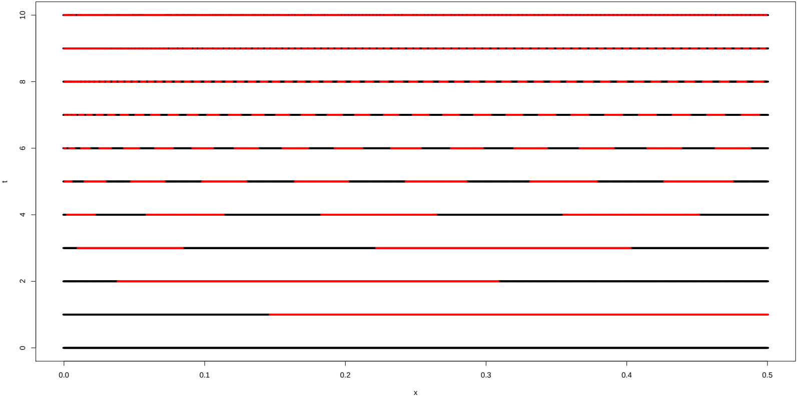
Chaos as a source of randomness

Black means $x \leq 0.5$, red means $x > 0.5$; here is $r = 1$



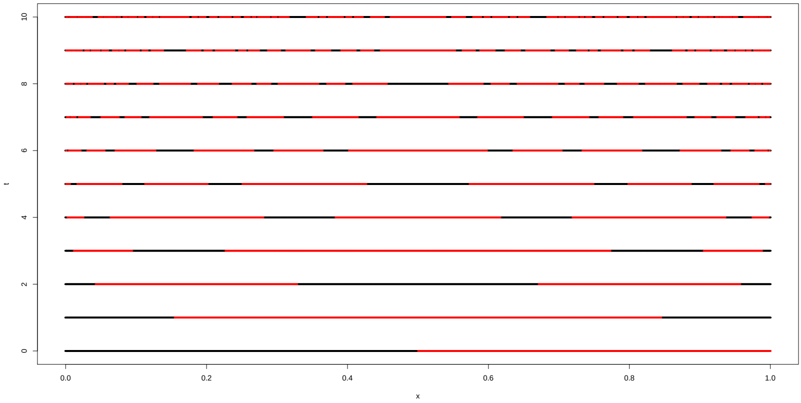
```
plot.little.line = function(center,width,height,...) {
  lines(c(center-width,center+width),c(height,height),...)
}

logistic.map.fates = function(iterations,n=1000,from=0,to=1,r=1,...) {
  x = seq(from=from,to=to,length.out=n)
  x.ic = x
  plot(NULL,NULL,xlim=c(from,to),ylim=c(0,iterations),xlab="x",
       ylab="t")
  iterate = 0
  while(iterate <= iterations) {
    blacks = x.ic[x <= 0.5]
    reds = x.ic[x > 0.5]
    num.blacks = length(blacks)
    num.reds = length(reds)
    sapply(blacks, plot.little.line, width=1/(2*n),height=iterate,
           col="black",...)
    sapply(reds, plot.little.line, width=1/(2*n),height=iterate,
           col="red",...)
    iterate = iterate+1
    x = logistic.map(x,r)
  }
}
```

zoom in on the left half

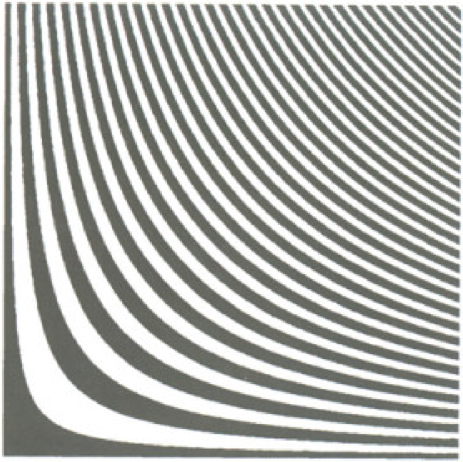


by $t = 10$ looks pretty much like whole thing
knowing initial condition helps you less and less as time goes on

and here is $r = 0.9571$ (to check this isn't just $r = 1$)



Compare to Keller's picture of coin tossing (via Guttorp):



Coin-tossing: very fine control of initial conditions needed at reasonable speeds
re-setting between tosses

Logistic map: crude control of initial conditions needed *at first* no degree of control keeps working

One way to get eventual independence is to work at this **coarse-grained** level