Chaos, Complexity, and Inference (36-462)

Lecture 15

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Estimating Heavy-Tailed Distributions

Maximum likelihood The good way to get power law parameter estimates

Log-log regression The bad way to get power law parameter estimates

Non-parametric density estimation Do you care if you *have* a power law?

Further reading: Clauset *et al.* (2007)

Maximum Likelihood

Start with the Pareto (continuous) case probability density:

$$p(x; \alpha, x_{\min}) = \frac{\alpha - 1}{x_{\min}} \left(\frac{x}{x_{\min}} \right)^{-\alpha}$$

Assuming IID samples, log-likelihood is easy

$$\mathcal{L}(\alpha, x_{\min}) = n \log \frac{\alpha - 1}{x_{\min}} - \alpha \sum_{i=1}^{n} \log \frac{x_i}{x_{\min}}$$

Take derivative and set equal to zero at the MLE:

$$\frac{\partial}{\partial\alpha}\mathcal{L} = \frac{n}{\alpha-1} - \sum_{i=1}^{n}\log x_i/x_{\mathrm{min}}$$

$$\hat{\alpha} = 1 + \frac{n}{\sum_{i=1}^{n}\log x_i/x_{\mathrm{min}}}$$

What about x_{min}? If we know that it's really a Pareto, then the MLE for x_{min} is $\min x_i$. Otherwise, see later.

Zipf or Zeta Distribution

Same story:

$$
\begin{aligned}
p(x;\alpha,x_{\min}) &= \frac{x^{-\alpha}}{\zeta(\alpha,x_{\min})} \\
\mathcal{L}(\alpha,x_{\min}) &= -n\log\zeta(\alpha,x_{\min}) - \alpha\sum_{i=1}^{n}\log x_i \\
\frac{\zeta'(\hat{\alpha},x_{\min})}{\zeta(\hat{\alpha},x_{\min})} &= -\frac{1}{n}\sum_{i=1}^{n}\log x_i
\end{aligned}
$$

In practice it's easier to just maximize $\mathcal{L}$ numerically than to solve that equation

When $x_{\min} > 6$ or so,

$$\hat{\alpha} \approx 1 + \frac{n}{\sum_{i=1}^{n} \log \frac{x_i}{x_{\min} - 0.5}}$$

Result due to M. E. J. Newman, see Clauset *et al.* (2007)

Properties of the MLE

1. Consistency: easiest to see for Pareto. By LLN,

$$\frac{1}{n}\sum_{i=1}^{n}\log x_i/x_{\min} \rightarrow \mathbf{E}\left[\log X/x_{\min}\right] = \frac{1}{\alpha-1}$$

so $\hat{\alpha} \rightarrow \alpha$; similarly for Zipf

2. Standard error

$$\mathrm{Var}\left[\hat{\alpha}\right] = \frac{(\alpha-1)^2}{n} + O(n^{-2})$$

Can plug in $\hat{\alpha}$, or do jack-knife or bootstrap

Nonparametric Bootstrap in One Slide

Wanted: sampling distribution of some estimator $\hat{G}$ of a functional G of a distribution F

Given: data $x_1, x_2, \ldots x_n$, all assumed IID from F

Procedure: draw n samples, *with replacement*, from data, giving $b_1, b_2, \ldots b_n$

Calculate $\hat{G}(b_1, \ldots b_n) = \hat{G}_b$

Repeat many times

Empirical distribution of $\hat{G}_b$ is about the sampling distribution of $\hat{G}$

Properties of the MLE (continued)

3. Asymptotically Gaussian and efficient:

$$\hat{\alpha} \rightsquigarrow \mathcal{N}(\alpha, \frac{(\alpha - 1)^2}{n})$$

and this is the fastest rate of convergence
4. (Pareto) If x_{min} is known or fixed, $(\hat{\alpha} - 1)/n$ has an inverse gamma distribution, which gives exact confidence intervals

Log-Log Regression

Recall that for a power law

$$\begin{aligned} F^{\uparrow}(x) = \Pr(X \geq x) &\propto x^{-(\alpha-1)} \\ \log F^{\uparrow}(x) &\propto C - (\alpha - 1) \log x \end{aligned}$$

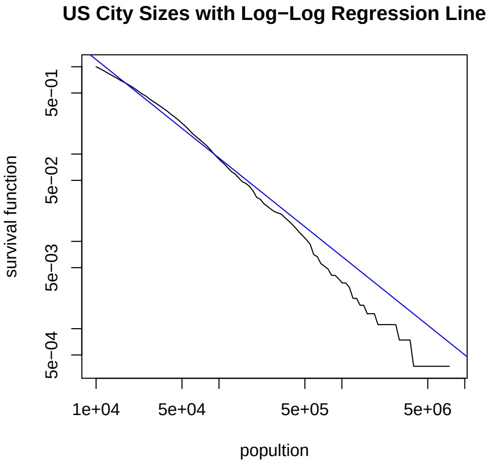
Empirical survival function:

$$\hat{F}_n^{\uparrow}(x) \equiv \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{[x,\infty)}(x_i)$$

As $n \rightarrow \infty$, $\hat{F}_n^{\uparrow}(x) \rightarrow F^{\uparrow}(x)$.
Estimate α by linearly regressing $\log \hat{F}_n^{\uparrow}(x)$ on $\log x$.

History

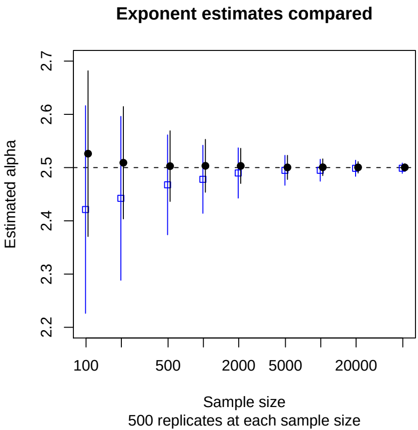
First real investigation of power law data came with Villfredo Pareto's work on economic inequality in 1890s
Used log-log regression
Taken up by Zipf in 1920s–1940s
Very widely used in physics, computer science, etc.


US City Sizes with Log−Log Regression Line
survival function
5e−01
5e−02
5e−03
5e−04
1e+04
5e+04
5e+05
5e+06
popultion

If the data really come from a power law, this is consistent:

$$\hat{\alpha}_{LLR} \rightarrow \alpha$$

but this doesn't say *how fast* it converges, and in fact the errors are large and persistent (compared to MLE)



Simulated from $\mathrm{Pareto}(2.5, 1)$; blue = regression, black = MLE (shifted a bit for clarity); mean $\hat{\alpha} \pm$ standard deviation

Why This Is Bad: Improperly Normalized

Notice that $F^{\uparrow}(x_{\min}) = 1$, so true log survival function crosses 0 at $x_{\min}$

But least-squares line does not do so in general! $\Rightarrow$ estimated function cannot be a probability distribution!

Could do constrained linear regression — but somehow you never see that

Why This Is Bad: Wrong Error Estimates

Usual formulas for standard errors in regression assume Gaussian noise

$$Y = \beta_0 + \beta_1 Z + \epsilon, \ \epsilon \sim \mathcal{N}(0, \sigma^2)$$

so using those formulas here means you're assuming *log-normal* noise for $\hat{F}_n^{\uparrow}$
and the central limit theorem says $\hat{F}_n^{\uparrow}$ has *Gaussian* noise
i.e., the usual formulas *do not apply* here
Can get error estimates (if you must) by bootstrap

Why This Is Bad: Lack of Power

People often point to a high R^2 for the regression as a sign that it must be right

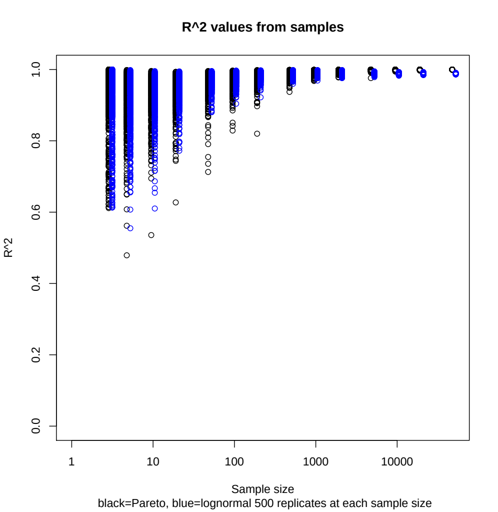
This is *always* foolish when it comes to regression

This is *especially* foolish here — distributions like log-normal have very high R^2 even with *infinite* samples

The R^2 test lacks **power** and **severity** against such alternatives

Example: Log-Log Regressions of Power Laws and Log-Normals

Simulated from $\mathrm{Pareto}(2.5, 5)$ and $\log \mathcal{N}(0.6626308, 0.65393343)$ — chosen to come close to the former. (Also, simulated values < 5 discarded.)
Did log-log regression for both, plot shows distribution of R^2 values from simulations.



Note: R^2 stabilizes at over 0.9 for the log-normal!

Log-Log Regression on the Histogram

Bad as log-log regression of the survival function is, it's still better than log-log regression of the histogram

1. Loss of information (not true with survival function)
2. Results depend on choice of bins for histogram
3. Even bigger normalization issues
4. Even worse errors — comparatively larger fluctuations, especially in the tail where they have the most leverage on the regression

There *may* be times when log-log regression on the survival function is reasonable (though I can't think of any); there are none when log-log regression on the histogram is

Conclusions about Log-Log Regression

1. Do not use it.
2. Do not believe people who use it.

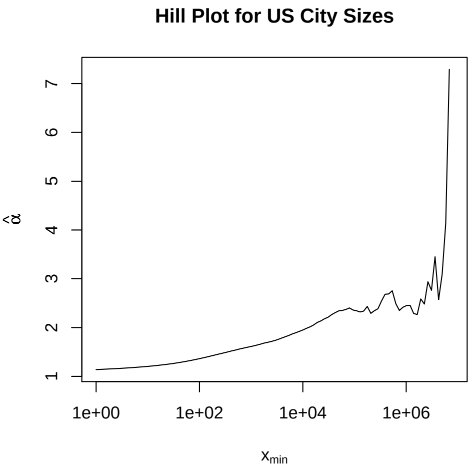
Estimating $x_{\min}$

Need to estimate $x_{\min}$
Simple for a pure power law: to maximize likelihood, $\widehat{x}_{\min} = \min x_i$
Not useful when it is only the *tail* which follows a power law

Hill Plots

One approach: try various $x_{\min}$, plot $\hat{\alpha}$ vs. $x_{\min}$, look for stable region
Called “Hill plot” after Hill (1975)
Also gives an idea of fragility of results

```
> hill.estimator <- function(xmin,data)
  {pareto.fit(data,xmin)$exponent}
> hill.plotter <- function(xmin,data)
  {sapply(xmin,hill.estimator,data=data)}
> curve(hill.plotter(x,cities),from=min(cities),
  to=max(cities),log="x",
  main="Hill Plot for US City Sizes",
  xlab=expression(x_min),
  ylab=expression(alpha))
```



Note log scale of horizontal axis

Kolmogorov-Smirnov Distance

Kolmogorov-Smirnov statistic: measure of distance between one-dimensional cumulative distribution functions

$$D_{KS}(F, G) = \sup_{x} |F(x) - G(x)|$$

Here look at

$$D_{KS}(x_{\min}) = \sup_{x \geq x_{\min}} |\hat{F}_n^{\uparrow}(x) - P(x; \hat{\alpha}, x_{\min})|$$

where $P(x; \hat{\alpha}, x_{\min})$ is the Pareto survival function we get by assuming a given $x_{\min}$ and estimating

Estimate $x_{\min}$ by Minimizing D_{KS}

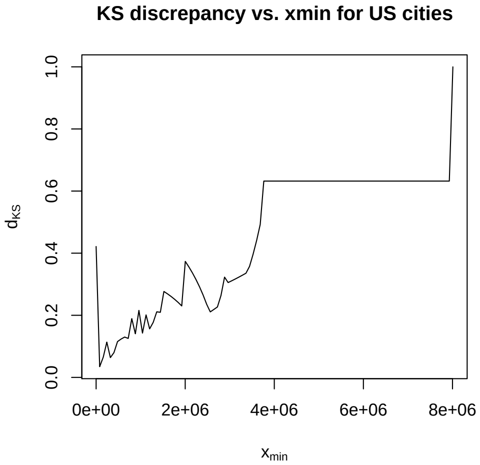
Pick the $x_{\min}$ where the distance between data and estimated distribution is smallest

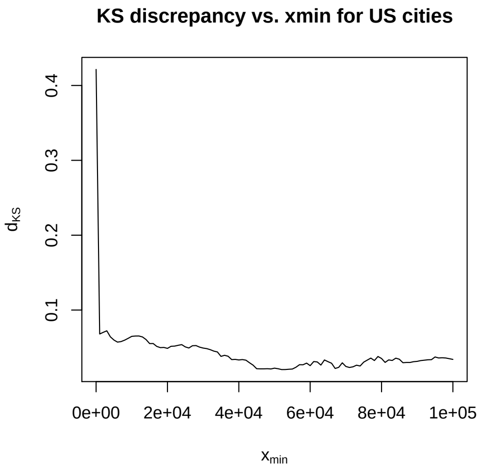
$$\widehat{x}_{\min} = \underset{x_{\min} \in x_i}{\operatorname{argmin}} D_{KS}(x_{\min})$$

Only considering actual data values is faster and seems to not miss anything

Another principled approach: BIC (Handcock and Jones, 2004), *but* we find that works slightly less well than minimum KS (Clauset *et al.*, 2007)

```
> ks.test.for.pareto <- function(threshold,data) {
    model <- pareto.fit(data,threshold)
    d <- ks.test(data[data>=threshold],ppareto,
      threshold=threshold,exponent=model$exponent)
   return(as.vector(d$statistic)) }
> ks.test.for.pareto.vectorized <- function(threshold,data)
  { sapply(threshold,ks.test.for.pareto,data=data) }
> curve(ks.test.for.pareto.vectorized(x,cities),
    from=min(cities),to=max(cities),
    xlab=expression(x[min]),ylab=expression(d[KS]),
    main="KS discrepancy vs. xmin for US cities")
```


KS discrepancy vs. xmin for US cities
1.0
0.8
0.6
0.4
0.2
0.0
dKS
0e+00
2e+06
4e+06
6e+06
8e+06
xmin



$\widehat{x}_{\min} = 5.246 \times 10^4$

Properties

1. In simulations, when there *is* a power law tail, this is good at finding it
2. When there isn't a distinct tail but there is an asymptotic exponent, choses $x_{\min}$ such that $\hat{\alpha}$ becomes right
3. Error estimates: bootstrap

Nonparametric Density Estimation as an Alternative

All of this is *assuming* a power-law tail, i.e., parametric form

Often this is neither justified nor important, but estimating the distribution is

Can then use non-parametric density estimation

Kernel Density Estimation in One Slide

Data $x_1, x_2, \ldots x_n$

$$\hat{p}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}K\left(\frac{x - x_i}{h_n}\right)$$

where **kernel** K has $K \geq 0$, $\int K(x)dx = 1$, $\int xK(x)dx = 0$, $0 < \int x^2 K(x)dx < \infty$
and **bandwidth** $h_n \to 0, nh_n \to \infty$ as $n \to \infty$
common choice of K: standard Gaussian density ϕ
$h_n = O(n^{-1/3})$ is best
R command: `density`

Ordinary non-parametric estimation works poorly with heavy-tailed data, since it generally produces light tails
Special methods exist, e.g.:

- transform data so $[0, \infty) \mapsto [0, 1]$ monotonically
- do ordinary density estimation on transformed data
- apply reverse transformation to estimated density

See Markovitch and Krieger (2000) or Markovich (2007) (harder to read)

Next time: how to tell the difference between power laws and other distributions

Clauset, Aaron, Cosma Rohilla Shalizi and M. E. J. Newman (2007). "Power-law distributions in empirical data." *SIAM Review*, **submitted**. URL http://arxiv.org/abs/0706.1062.

Handcock, Mark S. and James Holland Jones (2004). "Likelihood-based inference for stochastic models of sexual network formation." *Theoretical Population Biology*, **65**: 413–422. URL http://www.csss.washington.edu/Papers/wp29.pdf.

Hill, B. M. (1975). "A simple general approach to inference about the tail of a distribution." *Annals of Statistics*, **3**: 1163–1174. URL http://links.jstor.org/sici?sici=0090-5364%28197509%293%3A5%3C1163%3AASGATI%3E2.0.CO%3B2-Y.

Markovich, Natalia (2007). *Nonparametric Analysis of*

Univariate Heavy-Tailed Data: Research and Practice. New York: John Wiley.

Markovitch, Natalia M. and Udo R. Krieger (2000). "Nonparametric estimation of long-tailed density functions and its application to the analysis of World Wide Web traffic." *Performance Evaluation*, **42**: 205–222. doi:10.1016/S0166-5316(00)00031-6.