Chaos, Complexity, and Inference (36-462)

Lecture 25

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Inference for Network Models

Matched Random Networks

Hierarchical Random Graphs

Discriminating Network Growth Modes

General reading: Hunter *et al.* (2008)

Matched Random Networks

"So, you think you've found an interesting network structure, do you? Well isn't that *special*!"

Some kinds of network structure follow automatically from others

e.g., assortative $\Rightarrow$ reciprocal, cluster

Is what you are seeing an artifact or does it mean something?

Question of *what does a random network look like*?

But not just any random network, one that is *close* to yours

Basic Algorithm

Observe interesting feature X in your data graph g
Construct a distribution μ over random graphs G that matches g, but doesn't build in X
Draw many samples $G_1, G_2, \ldots G_m$ from μ
See how many of them have feature X

Simple Matched Random Networks

Erdős-Rényi networks are random...
Matching: same number of nodes and same density of edges $=$ expected degree
Very random... in fact, too random
In almost any situation, you know that your network doesn't look like *that*
It would be nice to match some more features than just the size and the density!

Exponential Random Graphs as Matched Random Networks

Pick your functionals on the network = sufficient statistics = T_i, $i \in 1:d$, observe values $t_i = T_i(g)$
Then (as discussed) $\widehat{\theta}_{MLE}$ solves

$$\mathbf{E}_{\widehat{\theta}_{MLE}}[T] = t$$

BUT $\mu = \mathrm{Pr}_{\widehat{\theta}_{MLE}}(G)$ also solves

$$\max H_{\mu}[G] \text{ s.t. } \mathbf{E}_{\mu}[T] = t$$

with H = Shannon entropy (Mandelbrot, 1962).

Maximizing likelihood in the exponential family maximizes entropy over all distributions

Why Should You Care about Maximum Shannon Entropy?

Some people see this as a self-justifying ideal

this is hard to take seriously

Gives distribution closest to independence under the constraint

(Amari, 2001)

but observed $\approx$ expected isn't a universal rule of inference!

unless observation is a big average!

Background problem: picking the right statistics to match

Fixing the Degree Distribution

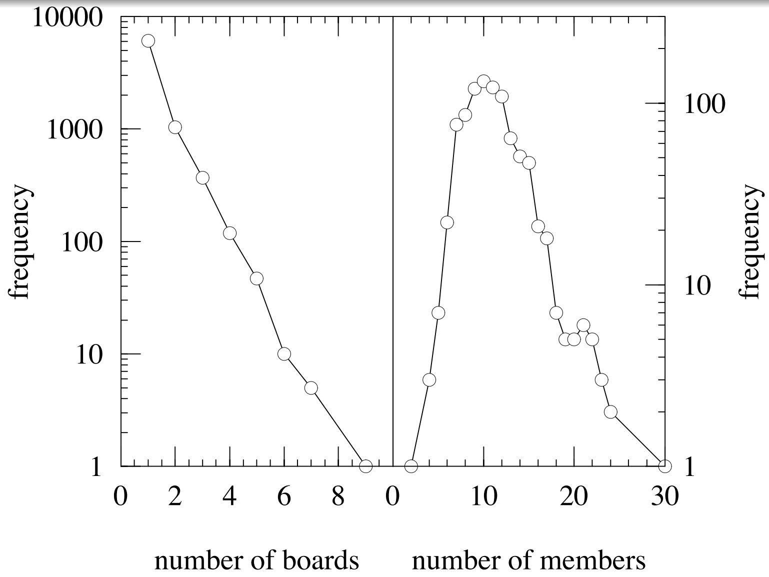
Newman *et al.* (2001)
Fix N
Generate N random numbers K_i from the empirical degree distribution — “stubs”
Choose pairs of free stubs uniformly at random; join them
Equally likely to produce any graph with that degree distribution
Must have even sum-of-degrees but this is not a big issue (if not even, discard and re-simulate)
Modifications required for directed graphs and bipartite graphs to handle summing-up constraints
if sums don’t match, pick one pair, discard their sizes, re-draw; repeat as needed

Example: Corporate Boards

Public corporations have **directors** who represent shareholders and (supposedly) pick the executives
Board members often sit on many boards
This effectively a coordinating mechanism

> *People of the same trade seldom meet together, even for merriment and diversion, but the conversation ends in a conspiracy against the public, or in some contrivance to raise prices.*

Also something that doesn't lead to very good decisions, but does shield rich people from market forces (Khurana, 2002)


10000
1000
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number of boards
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number of members
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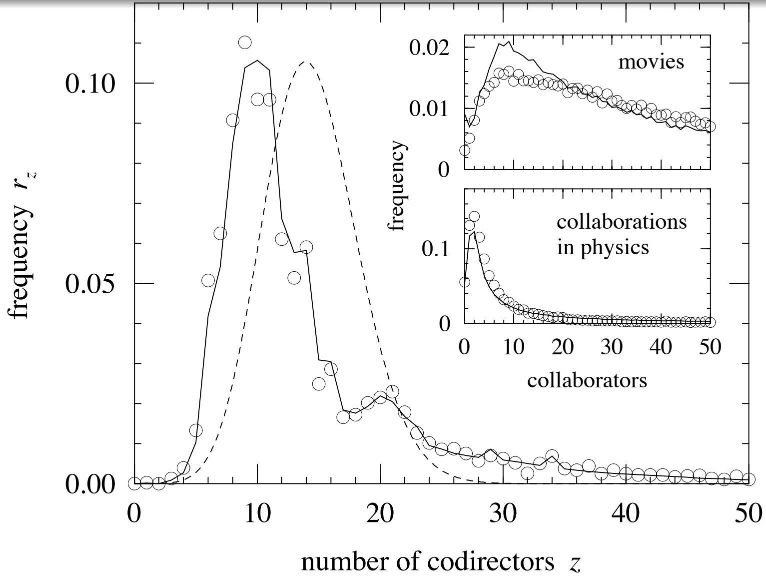
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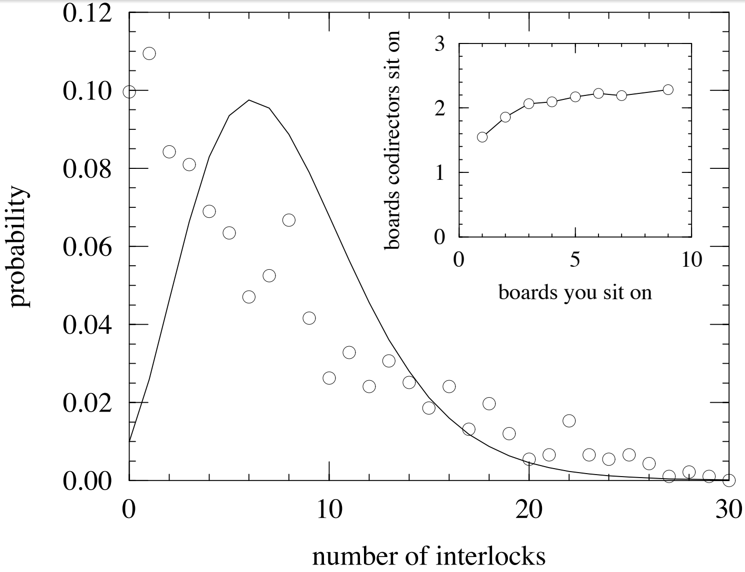

probability
number of interlocks
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
10
20
30
boards codirectors sit on
boards you sit on
3
2
1
0
0
5
10

Hierarchical Structure

Clauset *et al.* (2007)

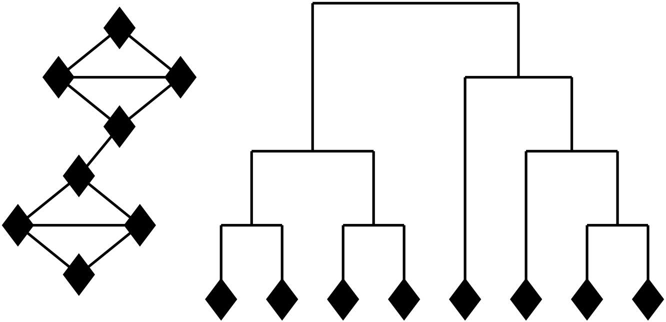
code: `http://www.santafe.edu/~aaronc/randomgraphs/`

Go back to the inhomogeneous Erdős-Rényi model

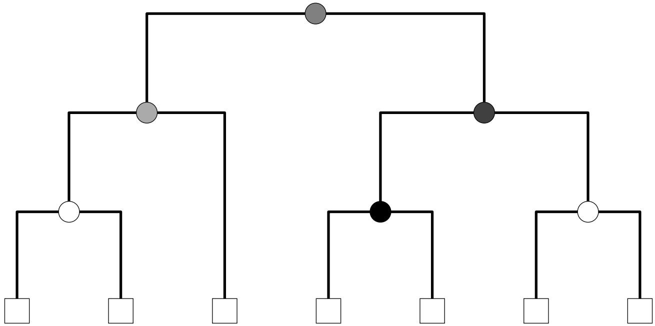
Completely arbitrary mixing matrix suffers a common problem of unconstrained maximum likelihood: over-fit by assigning probability 1 to data

here, 1 type per node

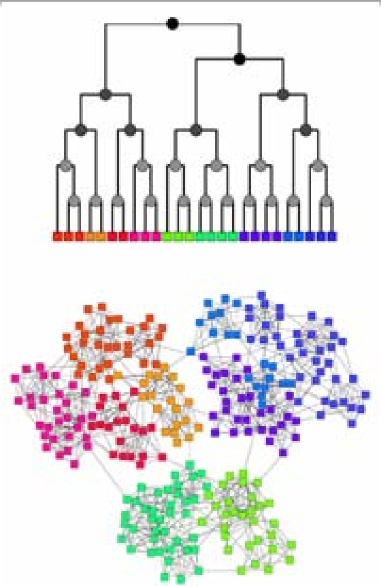
One constraint: hierarchy



Graph, hierarchy on nodes



Linking probability on tree for hierarchy



Probability of within-group linkage for group/type $i = \theta_i$
number of edges within group i, but not any of its sub-groups $= T_i$
number of nodes in left sub-tree, right sub-tree $= L_i, R_i$
Number of possible edges for group i is $L_i R_i$
so log-likelihood is

$$\mathcal{L}(\theta) = \sum_i T_i \log \theta_i + (L_i R_i - T_i) \log (1 - \theta_i)$$

Maximizing $\mathcal{L}$ assumes you know the tree! This is what we want to learn...

Could try all $\approx \sqrt{2}(2N)^{N-1}e^{-N}$ possible trees...

Parameter-counting penalties (like BIC) unhelpful since always $N-1$ parameters

What to do?

Model sampling and model averaging

Pick an initial tree τ however you like; maximized log-likelihood $= \mathcal{L}_\tau$

Randomly perturb it to make a new tree δ, log-likelihood $= \mathcal{L}_\delta$

Accept δ if $\mathcal{L}_\delta \geq \mathcal{L}_\tau$

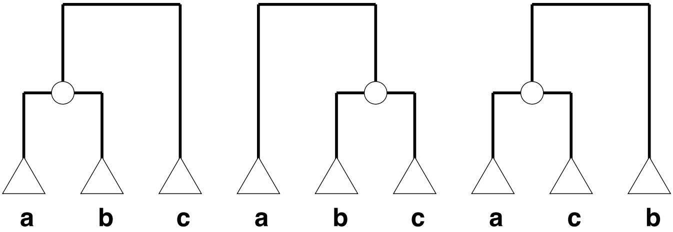
Otherwise, accept with probability equal to $e^{\mathcal{L}_\delta - \mathcal{L}_\tau}$

Gives a sample of trees where $\Pr(\tau) \propto e^{\mathcal{L}_\tau}$

Can average over trees, do weighted average (by likelihood), use common features of many trees...

Randomly perturbing trees

Pick random internal node in tree; has out-group and two in-groups; swap at random



this goes from any tree on N nodes to any other tree on N nodes

Discriminating Network Growth Modes

Middendorf *et al.* (2005)
Given: different models for how a network grew
Wanted: guess as to which one it was
Simulate many networks from each model
Train a **classifier** to reliably discriminate between them
Need **features** (sub-graph census) and **classifiers** (decision trees)
Validate the classifier by showing it has low error rates
Classifier may or may not look at features important in any one model

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