Chaos, Complexity, and Inference (36-462)

Lecture 26

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The Dollar Auction

Adaptive Behavior

The best introductory textbook on game theory is Gintis (2000). Less technical but good orientations: Poundstone (1992); Sigmund (1996); Slee (2006)

Games

Agents or players; "Nature" may be a player

Actions or moves

Pay-off reward or punishment for each player's action, *given* all the others' moves

Single-valued utilities are actually a very dubious assumption, on basic neurological grounds (McCulloch, 1945)

Game tree shows history of moves by all players to date

Strategy says which move to make at each node in game tree (possibly stochastic)

Best reply move/strategy which has highest pay-off *given* other player's moves/strategies

Equilibrium everyone plays best reply against everyone else

Dominated strategy Another strategy *always* does at least as well, and sometimes better

Minimax minimize the maximum harm suffered

"Rational" maximizing subjectively-expected payoffs, with personal, subjective probabilities updated by Bayes's rule; sometimes with extra assumption that subjective expectations are always objectively unbiased
"rationality" $\Rightarrow$ "elimination of dominated strategies"

Backwards induction Recursive elimination of dominated strategies

"Rationality" in action (1): the ultimatum game

"Rationality" in action (2): the prisoners' dilemma

Bounded rationality: not fully "rational", but uses an actual, implementable procedure to make decisions

Institutions simplify decisions so people can make them

Evolutionary Games

The classic work: Maynard Smith (1982)
Pay-offs are to strategies, which are the **replicators**; pay-off is now **fitness**, $f(s)$
Dynamics concern the population share or frequency $p(s)$ of the *replicators*
higher fitness $\Rightarrow$ bigger population share
implementations: genetics, imitation

Replicator Dynamics

Replicator equation:

$$\Delta p_t(s) = p_t(s)\left[f_t(s) - \sum_{s'} f_t(s')p_t(s')\right]$$

Note that

$$\sum_s \Delta p_t(s) = 0$$

so it stays normalized, $\sum_s p_t(s) = 1$
Defines a dynamical system which we can analyze like any other (Hofbauer and Sigmund, 1998)

ESS

Evolutionarily stable strategy: one which can't be invaded

For any $s' \neq s$, $\Delta p(s') < 0$ when $p(s) = 1 - \epsilon$, $p(s') = \epsilon$, ϵ sufficiently small

Not all equilbria are evolutionarily stable!

ESS = stable fixed point

Cooperation in Prisoner's Dilemma

Play with automata
Always cooperate vs. always defect: defect wins
Tit-for-tat
Tit-for-two-tats, etc.
Lindgren (1996) summarizes this line of thought
Spatial structure and spatial pattern formation: nice discussion in Sigmund (1996)

Reinforcement

Adaptation within individual, not across population
"weight" of action s

$$
\begin{aligned}
w_{t+1}(s) &= \alpha w_t(s) + (1-\alpha) f(s) \text{ if played } s \\
w_{t+1}(s) &= \alpha w_t(s) \text{ otherwise} \\
p_t(s) &= \frac{w_t(s)}{\sum_{s'} w_t(s')} \text{ or} \\
p_t(s) &= \frac{\exp w_t(s)}{\sum_{s'} \exp w_t(s')}
\end{aligned}
$$

can give *excellent* matches to human data (Salmon, 2001; Erev and Roth, 2001)

Note 1: “strategies” here do not have to be single moves but could be complicated
Note 2: can give similar dynamics to replicator equation (Börgers and Sarin, 1997; Borkar, 2002; Sato and Crutchfield, 2003)
Note 3: Many variants on shape of the reinforcement, precise learning dynamics, etc. — Sutton and Barto (1998) analyzes many versions used in AI and robotics

Convergence via Reinforcement

Polya's urn: start with one ball of each of k colors
X_t = color of ball drawn from urn, uniformly, at time t
put that ball back, and add another of that color

$$p_{t+1}(s) = \frac{p_t(s)(k+t) + \mathbf{1}_s(X_t)}{k+t+1}$$

Analysis of the urn model:

$$\begin{aligned}
\mathbf{E}\left[p_{t+1}(s)\right] &= \frac{k+t}{k+t+1}p_t(s) + \frac{1}{k+t+1}\mathbf{E}\left[\mathbf{1}_s(X_t)\right] \\
&= \frac{k+t}{k+t+1}p_t(s) + \frac{p_t(s)}{k+t+1} \\
&= \frac{k+t+1}{k+t+1}p_t(s) = p_t(s)
\end{aligned}$$

so $p_t(s)$ is a **martingale**
Bounded martingales converge almost surely $\Rightarrow$ p_t converges a.s.

General flavor of analysis holds much more generally: under reasonable conditions, if

$$\mathbf{E}\left[p_{t+1}\right] = f_n(p_t)$$

and

$$f_n \rightarrow f$$

then long-run behavior of p_t tracks that of the *deterministic* dynamical system

$$x_{t+1} = f(x_t)$$

(Arthur, 1994; Pemantle, 2007)

Networks and Games

Play with your neighbors: similar effects to spatial structure
actually, discrete space is a special case
Skyrms and Pemantle (2000); Pemantle and Skyrms (2004):
two decisions, who to play with and what strategy to follow
Reinforce ties tha lead to good pay-offs
Leads to *endogeneous* network formation

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