Chaos, Complexity, and Inference (36-462)

Lecture 3: Attractors, Some More Chaotic Systems, Mixing

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Long-run behavior and inference

In most systems, long-run behavior is dominated by **attractors**
Aspects:

Geometry Where do trajectories go?

Probability What is the long-run distribution?

Attractors: Geometry

Generalize from stable fixed points and limit cycles
Attractor $\approx$ stable invariant set which cannot be split into smaller stable invariant sets
"Attractor" because nearby points move closer towards it
Requires that the map compress state space (on balance; can expand in some directions)
so Arnold cat map has no attractors
Examples: The points in the bifurcation diagram are $\approx$ the attractors for different values of r
(Why only $\approx$?)
More examples: Henon map, Lorenz system

Henon Map

$$x_{t+1} = a - x_t^2 + by_t$$
$$y_{t+1} = x_t$$

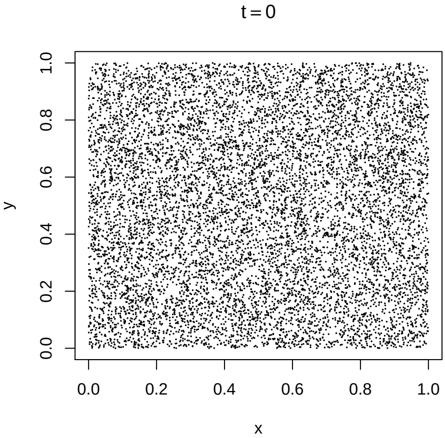
Two dimensions — y acts like a memory for x

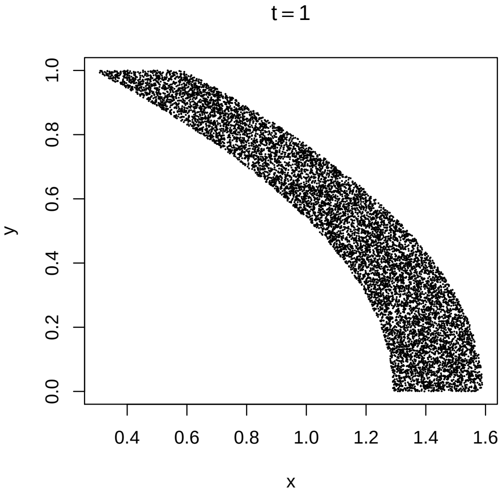
Important general fact about multidimensional dynamics

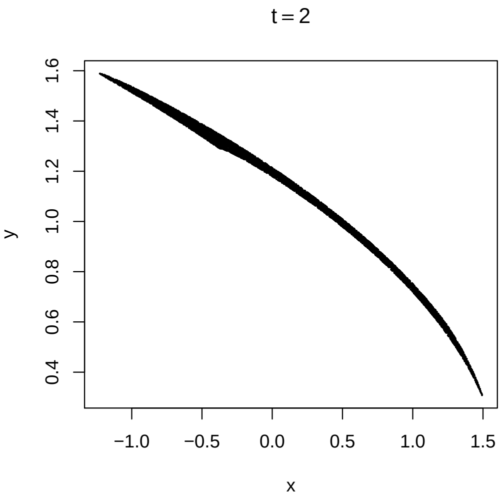
n dimension is equivalent to having a memory going back n time-steps

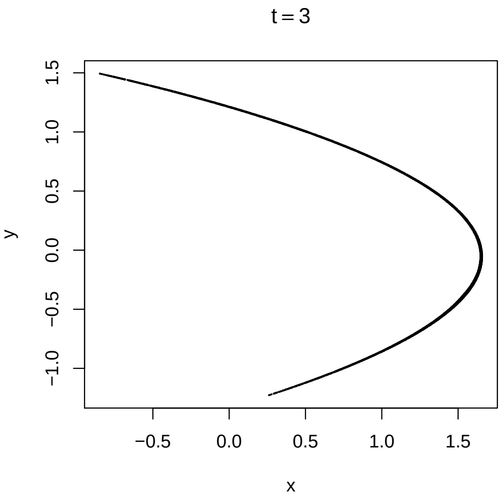
What happens with the Henon map?

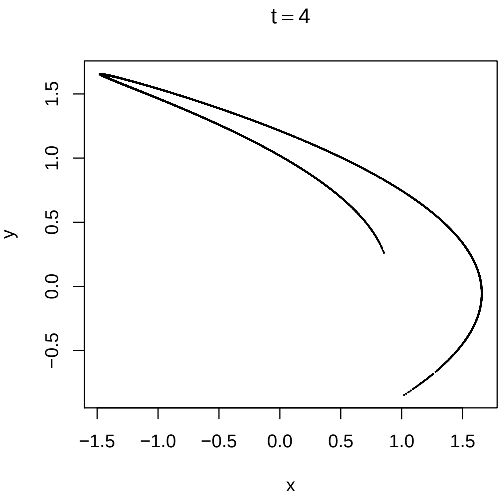
$a = 1.29$, $b = 0.3$

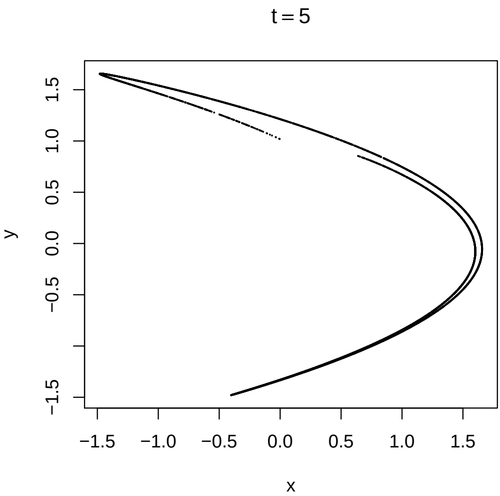


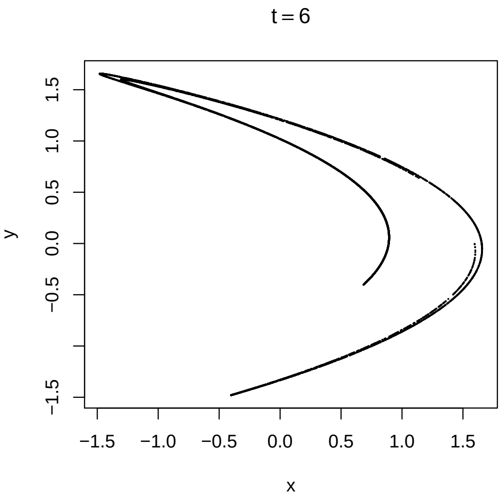

t = 1
1.0
0.8
0.6
0.4
0.2
0.0
y
0.4
0.6
0.8
1.0
1.2
1.4
1.6
x

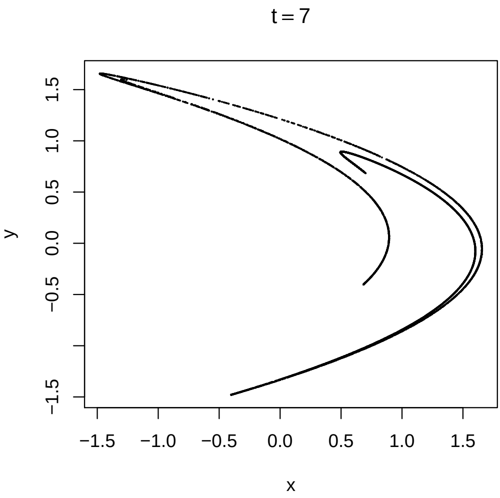

t = 2
y
x
1.6
1.4
1.2
1.0
0.8
0.6
0.4
−1.0
−0.5
0.0
0.5
1.0
1.5

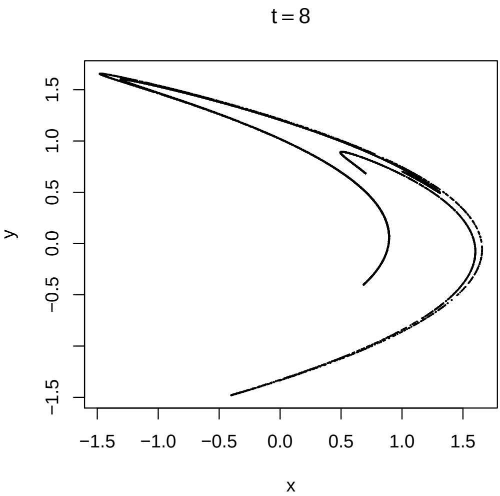

t = 3
y
1.5
1.0
0.5
0.0
−0.5
−1.0
x
−0.5
0.0
0.5
1.0
1.5

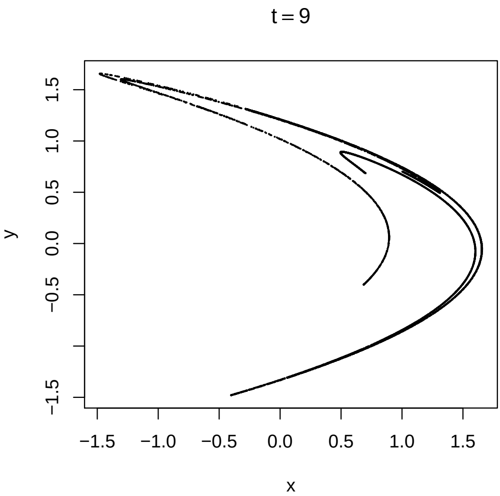

t = 4
y
1.5
1.0
0.5
0.0
−0.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

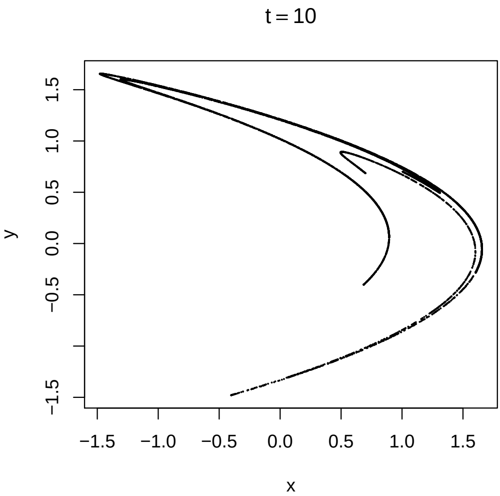

t = 5
1.5
1.0
0.5
0.0
−0.5
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

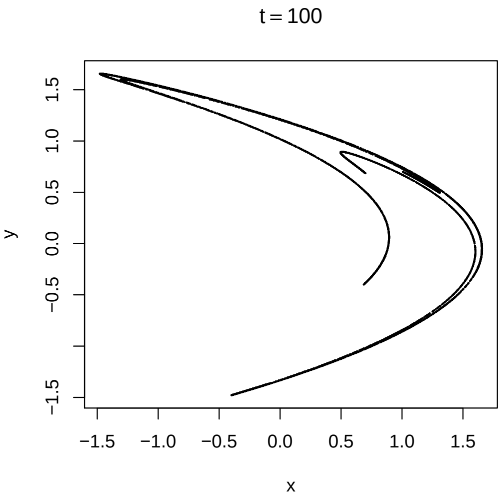

t = 6
1.5
1.0
0.5
0.0
−0.5
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x


t = 7
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

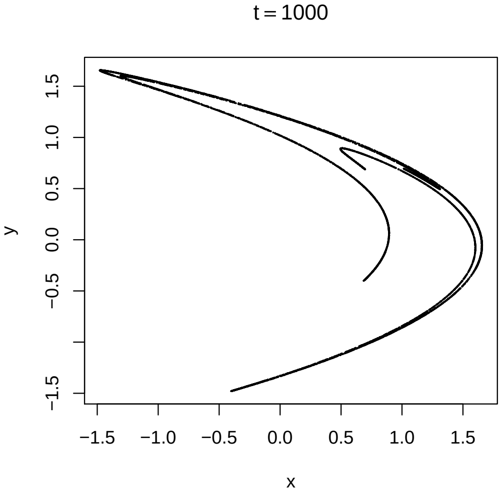

t = 8
1.5
1.0
0.5
0.0
−0.5
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

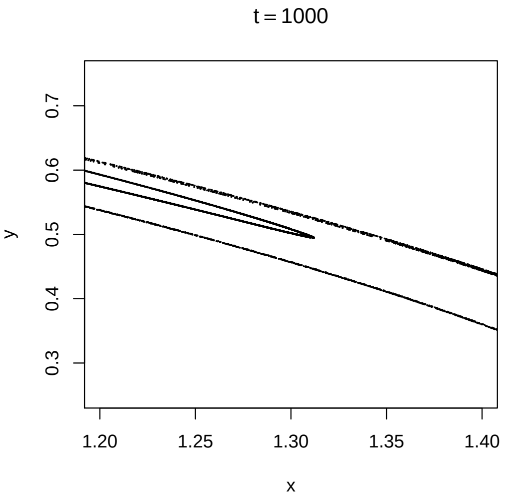

t = 9
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

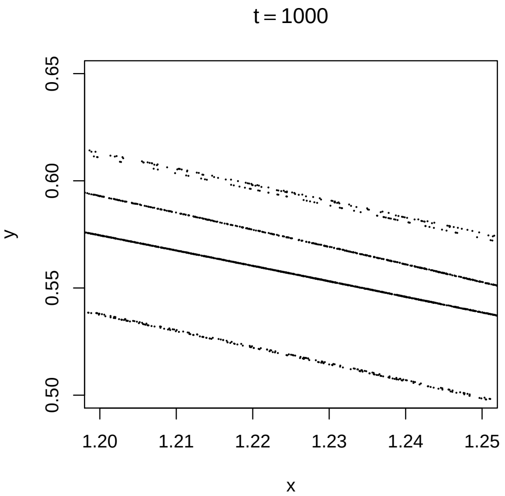

t = 10
y
1.5
1.0
0.5
0.0
−0.5
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x


t = 100
1.5
1.0
0.5
0.0
−0.5
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

take the attractor and zoom in




t = 1000
y
0.3
0.4
0.5
0.6
0.7
x
1.20
1.25
1.30
1.35
1.40


t = 1000
0.50
0.55
0.60
0.65
y
1.20
1.21
1.22
1.23
1.24
1.25
x

contraction ⇒ typically, attractors occupy a vanishingly small fraction of the state space

sometimes a well-behaved geometric object (points, curves)

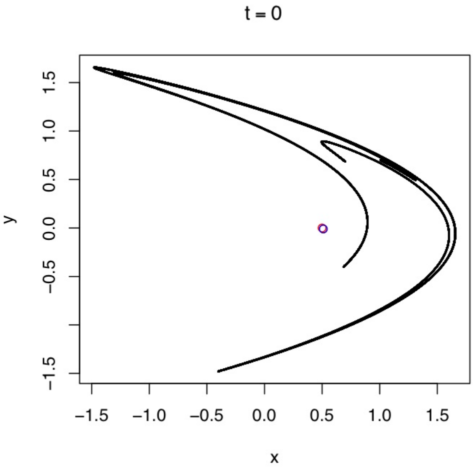
this attractor is “strange”: fractal (=fractional-dimensional), self-similar

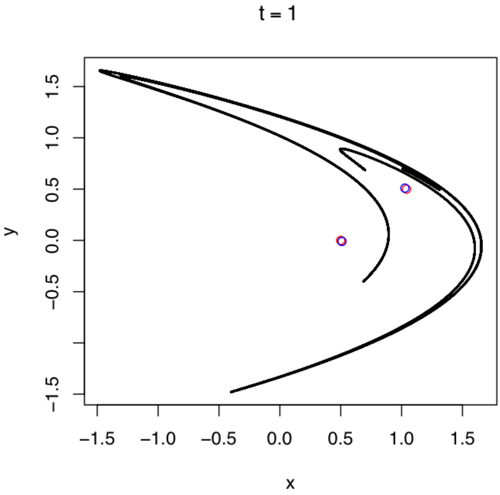
chaotic attractors are typically strange

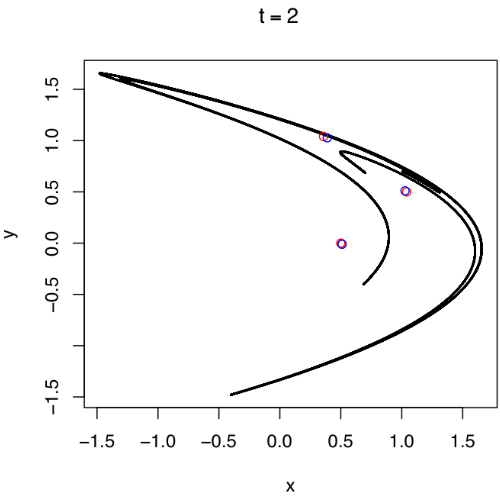
Stretch and Fold

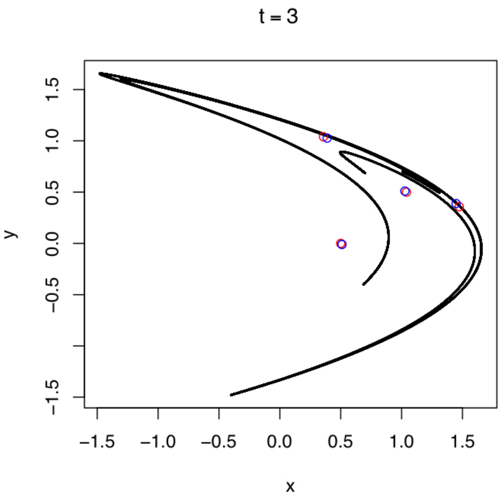
directions away from the attractor: stable; move back towards attractor
directions along the attractor: (possibly) unstable

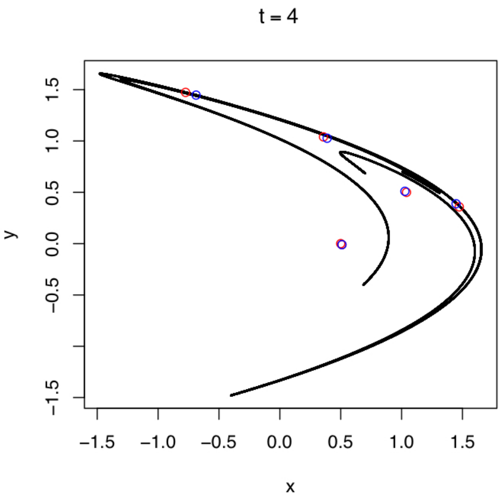
red: $(x_0, y_0) = (0.5, 0.0)$; blue: $(x_0, y_0) = (0.51, -0.01)$

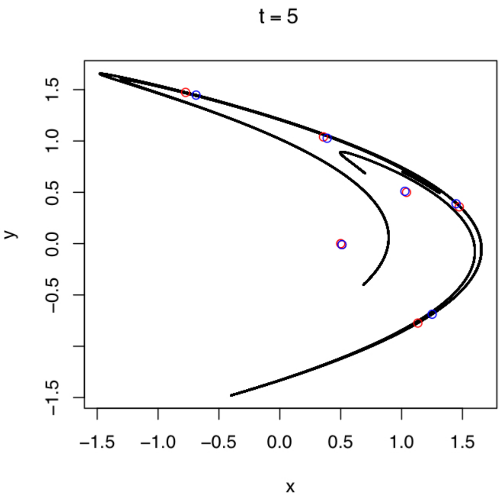


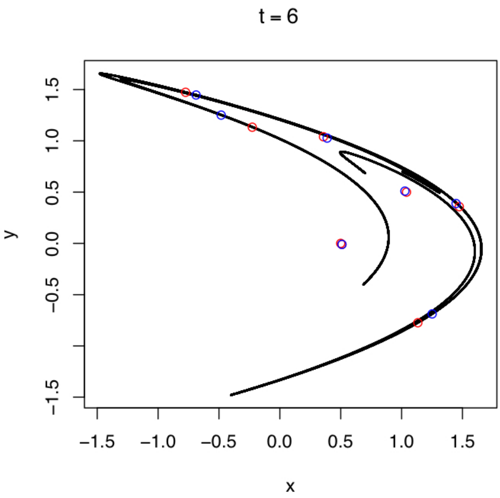

t = 1
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

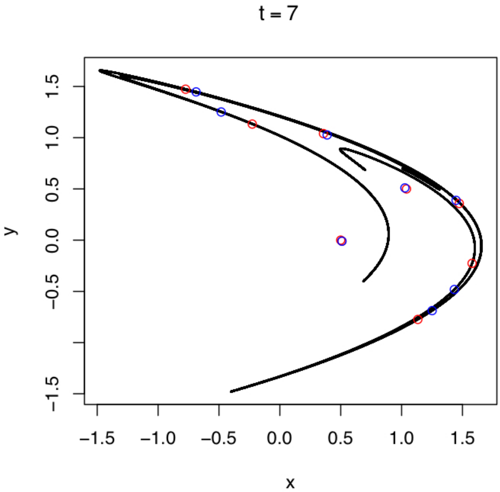

t = 2
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

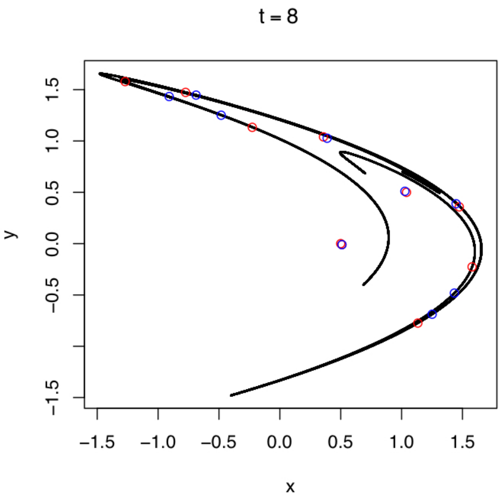

t = 3
y
1.5
1.0
0.5
0.0
−0.5
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

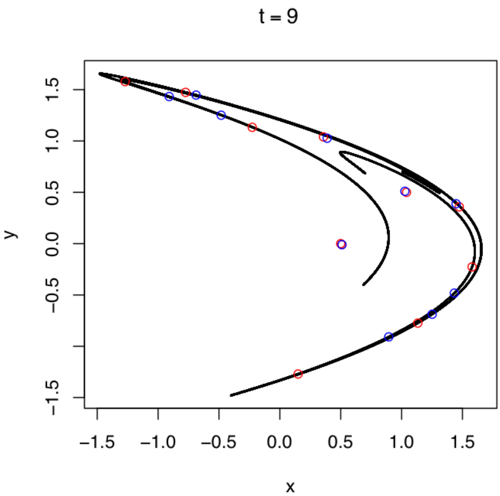

t = 4
y
1.5
1.0
0.5
0.0
−0.5
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

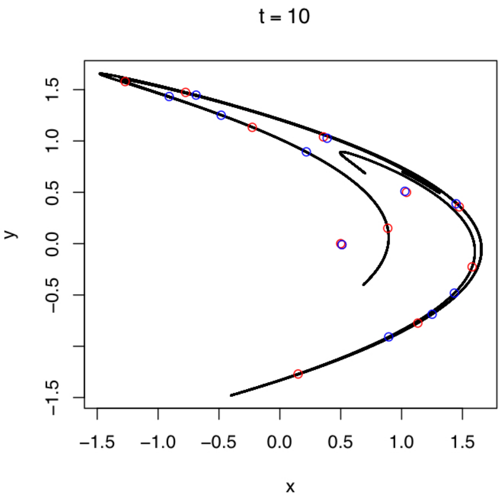

t = 5
y
1.5
1.0
0.5
0.0
−0.5
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

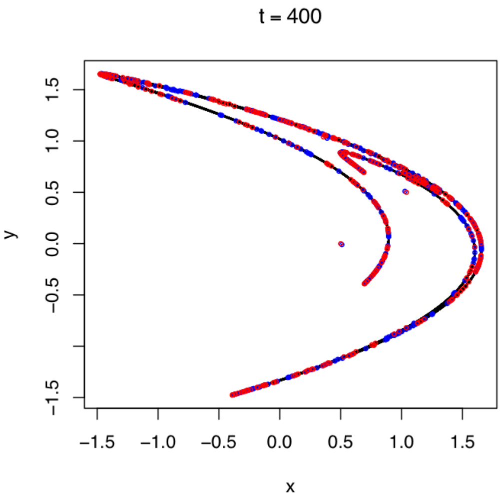

t = 6
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


t = 7
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


t = 8
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


t = 9
y
1.5
1.0
0.5
0.0
−0.5
−1.5
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


t = 10
y
1.5
1.0
0.5
0.0
−0.5
−1.5
−1.5 −1.0 −0.5 0.0 0.5 1.0 1.5
x

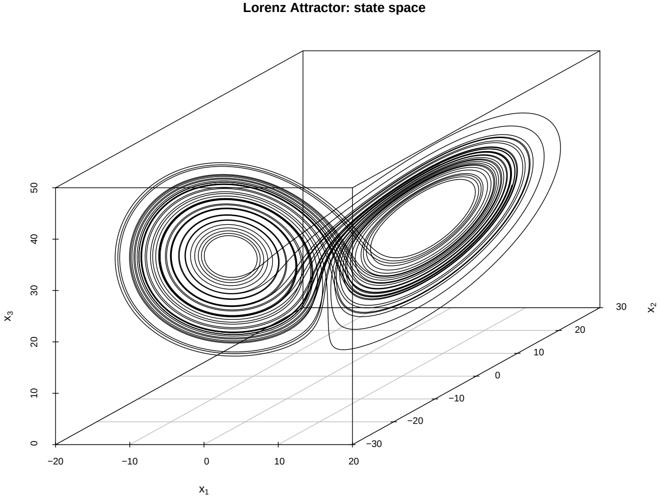

t = 400
y
1.5
1.0
0.5
0.0
−0.5
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

LORENZ SYSTEM

$$\frac{dx}{dt} = ay - ax$$
$$\frac{dy}{dt} = bx - y - xz$$
$$\frac{dz}{dt} = xy - cz$$

(more usual notation $a = \sigma$, $b = \rho$, $c = \beta$)
Crude approximation to tricky nonlinear model of fluid flow
Sensitive dependence on initial conditions discovered due to truncation

use $a = 10$, $b = 28$, $c = 8/3$



Lyapunov Exponents

p dimensional state-space
Stable and unstable directions locally at any point
Also rates of exponential contraction/expansion along them
so starting at x an initial separation of δ along some direction becomes a separation of $\delta e^{\Lambda(x)t}$ — not necessarily along that direction
There is always a most unstable direction $e_1(x)$, rate $\Lambda_1(x)$
Then a next most unstable direction $\perp e_1(x)$, rate $\Lambda_2(x)$
... finally a most stable direction $e_p(x)$, rate $\Lambda_p(x)$
Pick a trajectory; $\lambda_i \equiv$ time-average of $\Lambda_i(x)$
(How do we know that's well-defined?)
λ_i are the **Lyapunov exponents**
Practically, "chaos" means an attractor, and $\lambda_1 > 0$

Basin of attraction

all the initial conditions which converge on an attractor
We have only seen systems with one attractor but there can be many
another kind of unpredictability: points very near the boundary between 2 basins of attraction

Attractors: Probability

probability of states (invariant distributions)
probability of trajectory segments (correlations)
linked to geometry via instabilities in the attractor

Invariant distributions

Invariant distributions are (generally) confined to invariant sets — stable or unstable

Natural or **physical** invariant distributions are confined to attractors

Periodic attractors have uniform invariant distributions

Strange attractors generally have non-uniform invariant distributions

If multiple attractors then multiple physical invariant distributions

Volumes in state-space keep shrinking $\Rightarrow$ attractor is actually infinitely small compared to whole state space
(points vs. line, lines vs. plane, etc., or weird fractal shapes)
Invariant distributions are generally *not smooth at all*

They have to give probability 1 to tiny, weirdly-shaped sets, and probability 0 to most of the state space

Usually no simple parametric form ($r = 1$ logistic map is an exception)
Kernel density estimation on time series can work but too much smoothing is very misleading!

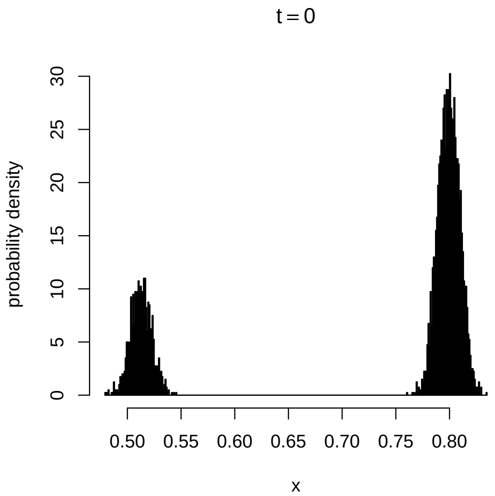
For a known 1D map, see Binder and Campos (1996)

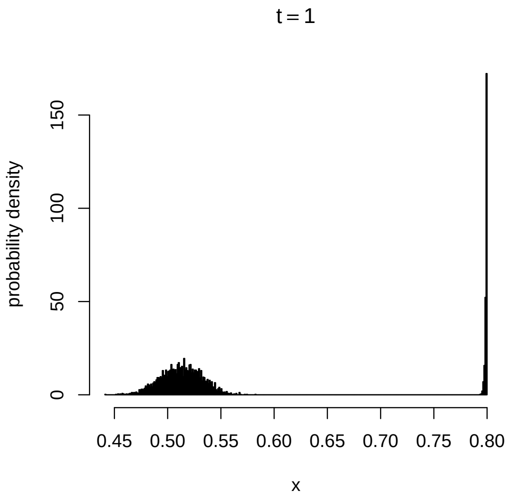
Convergence of Ensembles

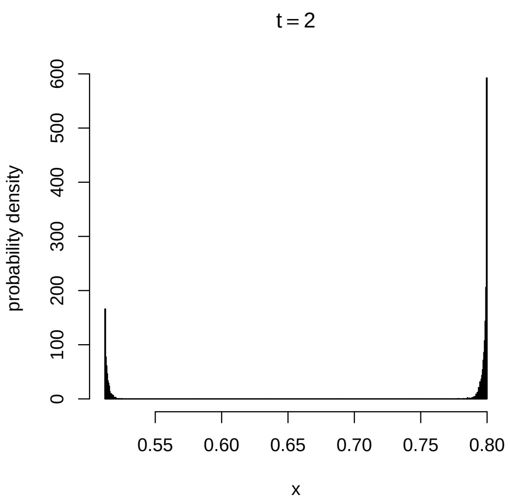
Ensembles converge to distributions on the attractor, not necessarily invariant ones
Example: stable 2-cycle; 3/4 of ensemble near one point, 1/4 near the other
converges to a distribution on the cycle
doesn't balance out

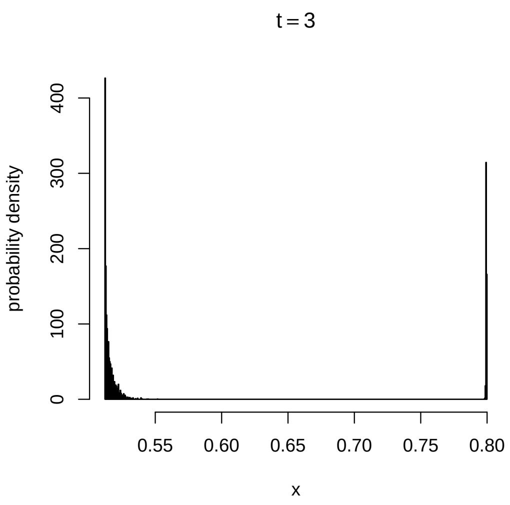
logistic map, $r = 0.8$, cycle from ≈ 0.51 to ≈ 0.80 and back

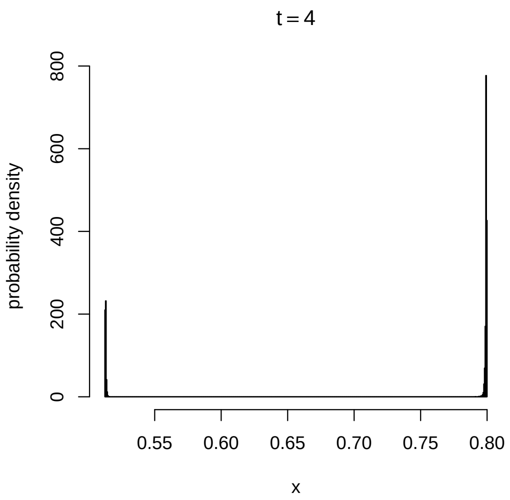
initial ensemble: two Gaussians centered at the cycle points, $\sigma = 0.01$, 4000 point, 3/4 near higher point

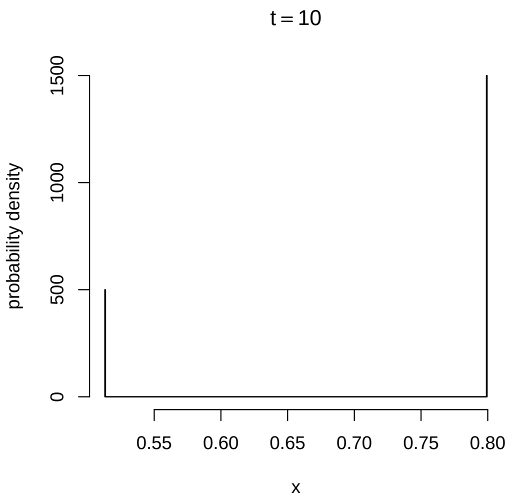

t = 0
probability density
0
5
10
15
20
25
30
0.50
0.55
0.60
0.65
0.70
0.75
0.80
x

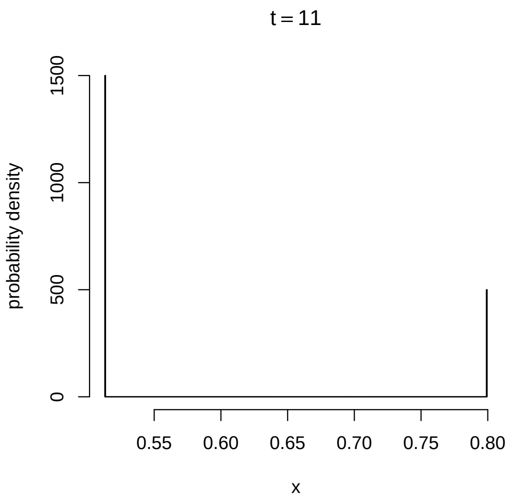

t = 1
probability density
0
50
100
150
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
x

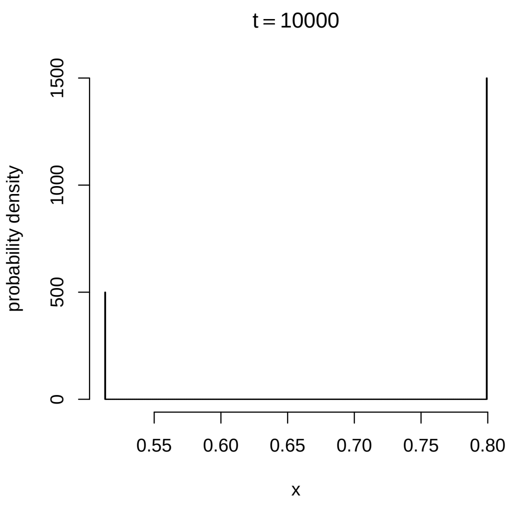

t = 2
probability density
0
100
200
300
400
500
600
0.55
0.60
0.65
0.70
0.75
0.80
x

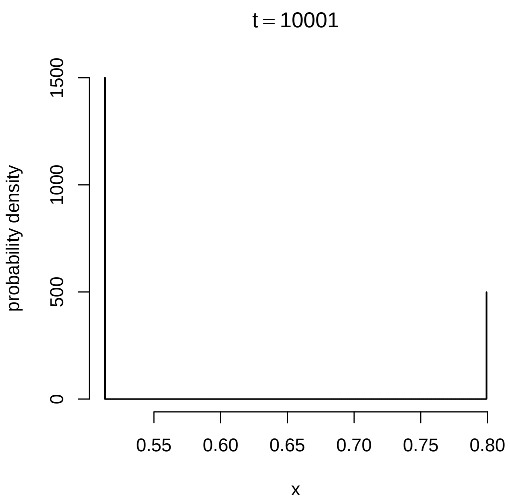

t = 3
probability density
0
100
200
300
400
0.55
0.60
0.65
0.70
0.75
0.80
x


t = 4
probability density
800
600
400
200
0
0.55
0.60
0.65
0.70
0.75
0.80
x


t = 10
probability density
0
500
1000
1500
0.55
0.60
0.65
0.70
0.75
0.80
x


t = 11
probability density
0
500
1000
1500
0.55
0.60
0.65
0.70
0.75
0.80
x


t = 10000
probability density
0
500
1000
1500
0.55
0.60
0.65
0.70
0.75
0.80
x


t = 10001
probability density
0
500
1000
1500
0.55
0.60
0.65
0.70
0.75
0.80
x

Ensemble: lopsided, alternates forever, oscillations never diminish

Individual trajectory's distribution: balanced, any initial bias diminishes steadily over time

Time-averaged ensemble looks like time-averaged single-trajectory distribution

Notice that there is no instability *within* the attractor here

If there was, we might expect ensembles to look more like individual trajectories

Mixing and decay of correlations

Mixing: As $\tau \rightarrow \infty$, X_t becomes independent of $X_{t+\tau}$
Equivalent to
Decay of correlations: for any reasonable functions g, h,

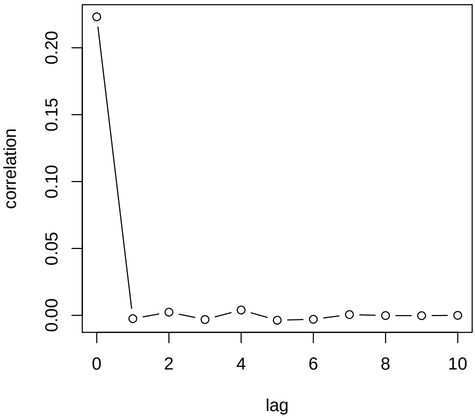
$$\mathrm{cov}[g(X_t), h(X_{t+\tau})] \xrightarrow[\tau\rightarrow\infty]{} 0$$

Equivalent to
Statistical stability: any initial ensemble converges to the natural invariant distribution

Probabilists: this is weak convergence

```
plot.decay.of.correlations = function(from=0,to=10,ts,...) {
  n=length(ts)
  rho = vector(mode="numeric",length=(to-from+1))
  for (t in from:to) {
    rho[t] = cor(cos(ts[1:(n-t)]),sin(ts[(1+t):n]))
  }
  plot(from:to,rho,xlim=c(from,to),xlab="lag",
       ylab="correlation",...)
}
```

Logistic map with $r = 1$, $g(x) = \cos x$, $h(x) = \sin x$, time series of length 10^5



Mixing $\Rightarrow$ ergodicity

Terminological confusion: an "ergodic Markov chain" is really a *mixing* Markov chain

Attractors can be non-mixing (e.g., periodic cycles)

Non-attractors can be mixing (e.g., cat map)

Chaotic attractors are (generally) mixing

(hierarchy of ergodic properties — see Mackey (1992); Lasota and Mackey (1994) for more)

Some uses of mixing

If $\tau \gg$ mixing time, $X_t, X_{t+\tau}, X_{t+2\tau}, \ldots$ are $\approx$ independent

Useful in building random number generators, Markov Chain Monte Carlo

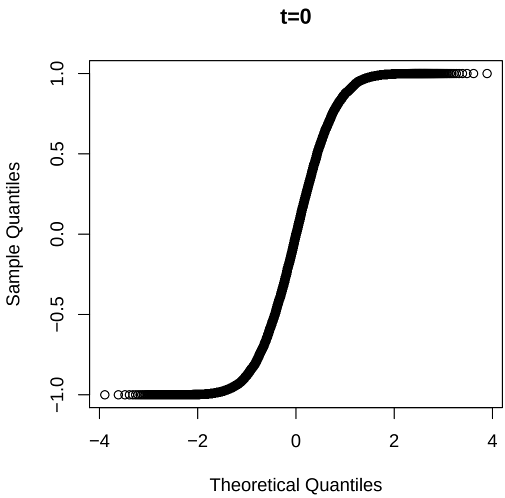
"Forgetting" of initial conditions

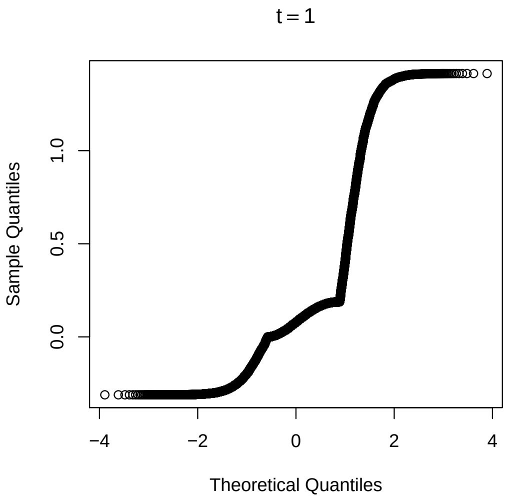
CLT: if correlations decay fast enough, time averages have a central limit theorem:

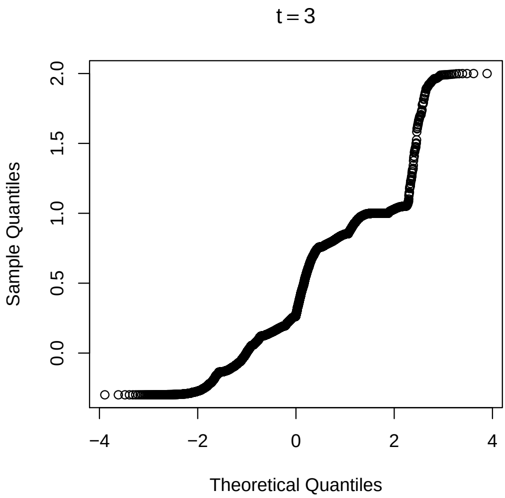
$$\sqrt{t}\left(\frac{1}{t}\sum_{i=1}^{t} f(X_i)\right) \rightsquigarrow \mathcal{N}(\mathbf{E}\left[f(X)\right], \sigma_f^2 \tau_{\mathrm{corr}})$$

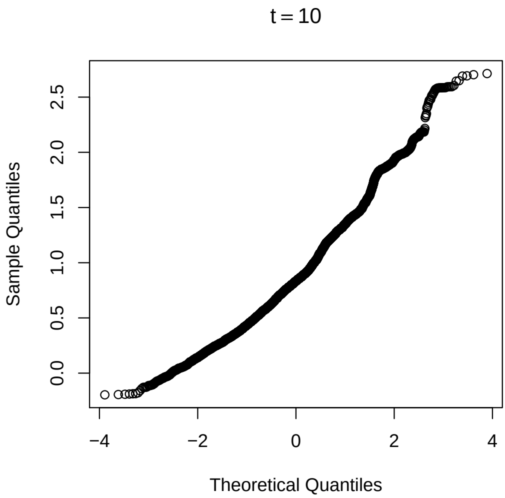
(Rosenblatt, 1956)

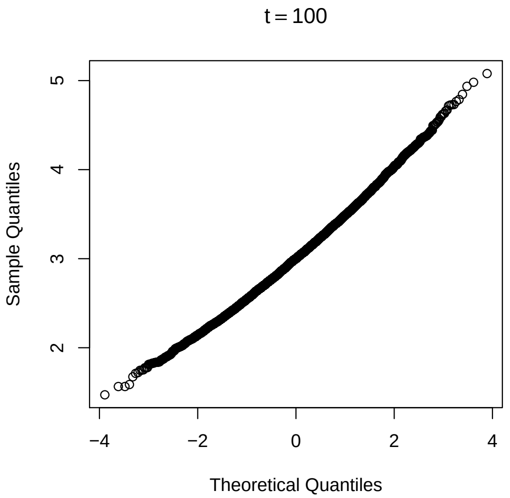
Illustrated for time-averages of $\cos 2\pi X_t$, X_t from $r = 1$ logistic map


t=0
1.0
0.5
0.0
−0.5
−1.0
Sample Quantiles
−4
−2
0
2
4
Theoretical Quantiles


t = 1
Sample Quantiles
0.0
0.5
1.0
−4
−2
0
2
4
Theoretical Quantiles


t = 3
Sample Quantiles
2.0
1.5
1.0
0.5
0.0
−4
−2
0
2
4
Theoretical Quantiles


t = 10
Sample Quantiles
0.0
0.5
1.0
1.5
2.0
2.5
−4
−2
0
2
4
Theoretical Quantiles


t = 100
Sample Quantiles
Theoretical Quantiles
2
3
4
5
−4
−2
0
2
4

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