Side-Note: Lyapunov Exponents

36-462, Spring 2009

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Abstract

This document elaborates a bit on the idea of Lyapunov exponents. The first part makes a couple of "it can be shown that claims". These are fleshed out in the second part, which uses linear algebra (eigenvalues and eigenvectors). For simplicity, everything is done in discrete time, for maps; continuous time, with flows, is basically similar but needs more notation.

1 The Basic Idea

Pick your favorite state x and update rule Φ. In general x is a vector in some d-dimensional space;the components are $x_1, x_2, \ldots x_d$. Imagine a small perturbation to this state, δ — again another vector. What happens to the perturbed initial condition under the update rule? Use Taylor expansion:

$$\Phi(x+\delta) = \Phi(x) + \delta\Phi'(x) + \text{small}$$

Here Φ' is the matrix of derivatives of Φ:

$$\Phi'_{ij} = \frac{\partial \Phi_i}{\partial x_j}$$

So

$$\Phi(x+\delta) - \Phi(x) \approx \delta\Phi'(x)$$

So the distance between the iterates of x and $x+\delta$ is proportional to the size of the perturbation, $\|\delta\|$ (at least if $\|\delta\|$ is small enough for Taylor expansion to work), but the constant of proportionality depends on the *direction* of the perturbation. That is,

$$\frac{\|\Phi(x+\delta) - \Phi(x)\|}{\|\delta\|} = \left\| \frac{\delta}{\|\delta\|} \Phi'(x) \right\| \tag{1}$$

where we can see that the right-hand side depends only on the normalized perturbation vector, i.e., the direction of δ.

It can be shown that there is some direction, call it $e_1(x)$, which maximizes (1). Call this maximum value $A_1(x)$. There is a $d-1$ dimensional space of directions which are all perpendicular to e_1. Among these, one of *them* maximizes

(1). Call this direction e_2, and its value $A_2(x)$. Proceeding in this way, one gets a sequence of d different vectors, all perpendicular to each other, $e_i \perp e_j$, with a corresponding sequence of magnification values, A_i. It can be shown that $e_d(x)$ is the direction in which (1) is smallest.

It can further be shown that if δ lies along the direction e_i, then $\Phi(x+\delta) - \Phi(x)$ also lies along that direction. So under repeated applications of the map, we would see

$$\frac{\|\Phi^n(x+\delta) - \Phi^n(x)\|}{\|\delta\|} = (A_i(x))^n = e^{n \log A_i(x)}$$

Let's abbreviate $\log A_i$ by Λ_i. The set of d numbers $\Lambda_1(x), \Lambda_2(x), \ldots \Lambda_d(x)$ are the **local Lyapunov exponents** at x. The directions e_i for which $\Lambda_i > 0$ are **locally unstable directions**, the directions for which $\Lambda_i < 0$ are **locally stable**, and the ones for which $\Lambda_i = 0$ are **neutral**.

Now $\Lambda_i(x)$ is a function of the location in the state space. One can calculate a time-average along a trajectory:

$$\frac{1}{T}\sum_{n=1}^{T} \Lambda_i(x_n)$$

If the dynamics are ergodic, this will converge to an invariant value, which is

$$\lambda_i \equiv \int \Lambda_i(x)\rho(x)dx$$

$\rho(x)$ being the invariant distribution. The expectation values λ_i are the **global Lyapunov exponents**; without any modifier, by default "Lyapunov exponent" refers to these global values.

2 The Bits with Linear Algebra

Φ takes a d-dimensional vector as input and gives a d-dimensional vector as output, so Φ' is a square matrix of derivatives. This means it has d eigenvectors and eigenvalues, which are orthogonal. That is, there are d vectors e_i, with $\|e_i\| = 1$ and $e_i \perp e_j$, such that

$$e_i\Phi' = A_i e_i$$

the scalar numbers A_i being the eigenvalues.

Let's try to find the *direction* that maximizes $\|\delta\Phi'\|$. This means fixing $\|\delta\| = 1$ and trying different directions. In other words, we are trying to maximize

$$(\delta\Phi') \cdot (\delta\Phi')$$

subject to the constraint

$$\delta \cdot \delta = 1$$

Introduce a Lagrange multiplier[1] μ to enforce the constraint:

$$\mathcal{L} = (\delta\Phi') \cdot (\delta\Phi') - \mu(\delta \cdot \delta - 1)$$

and maximize $\mathcal{L}$ with respect to both the δ and μ:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \delta} &= 2\delta\Phi' - 2\mu\delta = 0 \\ \frac{\partial \mathcal{L}}{\partial \mu} &= \delta \cdot \delta - 1 = 0 \end{aligned}$$

The first of these equations tells us that δ must be one of the eigenvectors of Φ', and that the Lagrange multiplier μ must be one of the eigenvalues. (The second equation just reminds us that δ must be normalized.) So the direction in which perturbations are most amplified is the eigenvalue with the largest eigenvector, and so on down the line to the eigenvector with the smallest eigenvalue.[2]

The eigenvectors form a basis for the space, so any vector can be written in terms of the eigenvectors:

$$\delta = \sum_{i=1}^{d} (\delta \cdot e_i) e_i$$

This means that

$$\delta\Phi' = \sum_{i=1}^{d} (\delta \cdot e_i) A_i e_i$$

so if we want to know what happens to an arbitrary perturbation δ, we just have to decompose it into the eigenvectors.

References

[1] Klein, Dan (2001). "Lagrange Multipliers without Permanent Scarring." Online tutorial. URL `http://dbpubs.stanford.edu:8091/~klein/lagrange-multipliers.pdf`.

[1] If you do not know what Lagrange multipliers are, read [1].

[2] Those of you who took 36-350 last semester should going "Stop, I've heard this one before" by now.