Side-Note: "Smooth Change of Coordinates" in Attractor Reconstruction

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Abstract

In the lecture, I used the phrase "smooth change of coordinates" a lot when talking about attractor reconstruction. As in, the geometry-from-a-time-series approach doesn't recover the actual state space, but it does recover something which is equivalent to it, up to a smooth change of coordinates. This (optional) note explains in more detail what that means.

In lecture, I claimed that attractor reconstruction lets us identify the underlying state-space of a dynamical system, "up to a smooth change of coordinates". What does this mean?

I mean something like going from using Cartesian (rectangular) coordinates to represent points on the plane to using polar coordinates. The points are the same, the geometric relationships between them are the same, but our numerical representation of them has changed. The point whose Cartesian coordinates are $(1, 1)$ has polar coordinates $(\sqrt{2}, \pi/4)$, but it's the same point.) If we try to write down equations of motion in terms of the individual coordinates, we'd need *different equations* in the two coordinate systems, even though they'd be representing the *same* dynamics.

EXERCISE: Re-write the Hénon map in polar coordinates.

There are two ways to go here; we can call them **abstraction** and **translation**.

The route of **abstraction** is to try to find a **coordinate-free** representation of the dynamics. This is the route that lead to vector calculus: ∇f always means the *same* vector field, whether you cash it out in Cartesian coordinates,

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

or in polar coordinates,

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{\partial f}{\rho \partial \theta}\hat{\theta}$$

The point of notation like ∇f is to give us rules for reasoning about this object in a way which abstracts from — that is, hides the details of — particular coordinate systems.[1] We will return to this idea of abstraction, but it's harder to pursue here.

The other route is that of **translation**. We stick with coordinate-dependent representations of the dynamics, but recognize that some of these are just different representations of the same thing, and study the transformations which take one representation into another — i.e., the coordinate changes which leave the dynamics alone. This route has also been followed by a lot of mathematicians. Unfortunately, they have created a truly ugly jargon to go with it.

A **morphism** is a mapping or transformation which preserves some structure or property we care about.[2] Here the structure in question is the relationship between states at time t and states at time $t+1$. To be a little more concrete, suppose X is our state variable and the update rule is Φ, so $X_{t+1} = \Phi(X_t)$. A function G of X is a morphism for this dynamical system when $G(X_{t+1}) = \Psi(G(X_t))$ for some map Ψ. That is, $G(\Phi(X_t)) = \Psi(G(X_t))$ — we can either apply the change of coordinates G first, and follow a new update rule Ψ, or we can apply the old update rule Φ and then change coordinates.

A morphism G becomes an **isomorphism**[3] if it is invertible, and the inverse G^{-1} also preserves the structure. That is, G^{-1} exists, and $\Psi(Y_t) = G(\Phi(G^{-1}(Y_t)))$. With a morphism, we can use either the old or the new coordinates to predict what will happen in the new coordinates, but we can't necessarily use the new coordinates to predict the old ones. With an isomorphism, we can go back and forth between the two coordinate systems at will. Invariant sets, like fixed points or periodic cycles, are "preserved" under isomorphisms (e.g., $\Phi(x) = x$ if and only if $\Psi(G(x)) = G(x)$, and similarly for periodic cycles).

An isomoprhism G becomes a **homeomorphism**[4] when it is continuous, and its inverse G^{-1} is also continuous. This is useful because mere isomorphisms can be very strange, but the continuity of homeomorphisms rules out some pathological cases. For example, there are one-dimensional maps which are isomorphic to some two-dimensional maps, but dimension is "preserved under homeomorphism". However, there can still be weird singularities in a homeomorphism, and they don't necessarily preserve the more quantitative properties of dynamical systems, like Lyapunov exponents.

A homeomorphism G becomes a **diffeomorphism** when its derivatives $\mathrm{D}c$ exist and are continuous, and G^{-1} also has continuous derivatives.[5] This is what I mean by a "smooth change of coordinates". Because the derivatives

[1] Follow this idea far enough and you wind up with modern differential geometry [5], and dynamical systems on manifolds [2]. This is actually very useful in advanced statistical theory, under the name of "information geometry" [1, 3, 4].

[2] The word is from the ancient Greek word *morphe*, meaning "shape".

[3] *Iso-* = "equal"

[4] *Homeo-* = "similar"

[5] One some-times useful bit of notation is to say that continuous functions belong to the class $\mathcal{C}^0$, functions with a continuous first derivative belong to $\mathcal{C}^1$, those with continuous second derivatives to $\mathcal{C}^2$, and so on to $\mathcal{C}^\infty$, where derivatives of all orders exist and are continuous. Then one says that a homeomorphism is an isomorphism where G and G^{-1} are both in $\mathcal{C}^0$, a diffeomorphism is when they are both in $\mathcal{C}^1$, etc. There is no special name, so far as I know, for isomorphisms in $\mathcal{C}^2$ or higher.

of G exist and are well-behaved, the quantitative properties of the dynamics which depend on the derivatives of the update rule, like the stability criteria and the Lyapunov exponents, are also preserved by diffeomorphisms.

EXERCISE: Use the fact that $\Psi(u) = G(\Phi(G^{-1}(u)))$ to Taylor-expand Ψ around some point $G(x)$. How does this compare to the Taylor expansion of Φ around x?

Example: The Hénon and Logistic Maps The logistic map is diffeomorphic to the Hénon map with $b = 0$. In fact, the diffeomorphism is a linear change of coordinates.

Remember that the update rule for the Hénon map is

$$h_{n+1} = a - h_n^2 \tag{1}$$

(because $b = 0$, we can ignore the second state coordinate), while the update rule for the logistic map is

$$l_{n+1} = 4rl_n(1 - l_n) \tag{2}$$

Set

$$h_n = c + dl_n \tag{3}$$

(and similarly for h_{n+1}). Then, substituting Eqs. 3 and 2 into Eq. 1 above, we have

$$a - (c + dl_n)^2 = c + d(4rl_n(1 - l_n)) \tag{4}$$

EXERCISE: solve this for a, c, d in terms of r. (*Hint*: each power of l_n must have the same coefficients on both sides of the equation. [Why?])

Example: the logistic map and the tent map The **tent map** is another map from the unit interval $[0, 1]$ into itself. The update rule is

$$t_{n+1} = \begin{cases} 2t_n & 0 \leq t_n \leq 1/2 \\ 2(1 - t_n) & 1/2 \leq t_n \leq 1 \end{cases} \tag{5}$$

(Why is it called the "tent" map?). This is diffeomorphic to the logistic map with $r = 1$, and the diffeomorphism is

$$l_n = \sin^2 \frac{\pi t_n}{2} \tag{6}$$

EXERCISE: convince yourself that this actually is a diffeomorphism.

As the last example suggests, if one is given two maps and asked to find a diffeomorphism between them, things can get ugly. But verifying that a given coordinate-change G is a diffeomorphism between two given maps is fairly straightforward, as is finding the Ψ which comes from combining a given G with a given Φ.

Attractor Reconstruction

In attractor reconstruction, the k-dimensional vector of time-delayed measurements R_t is diffeomorphic to the original state X_t. This means that $R_t = G(X_t)$ for some function G, and that there is some mapping in $\mathbb{R}^k$, call it Ψ, such that

$$R_{t+1} = \Psi(R_t)$$

while

$$G(\Phi(X_t)) = \Psi(G(X_t))$$

It is not possible to identify the original state-space X any more precisely than this. The reason is that "diffeomorphisms are closed under composition" — doing two different smooth changes of coordinates in a row is the same as doing one, direct change of coordinates. So if R is diffeomorphic to X, and X is diffeomorphic to Y, then R is also diffeomorphic to Y. In terms of the examples, we'll get the same reconstruction whether we start with the logistic map, or the tent map, or the $b = 0$ Hénon map.

For purely predictive purposes, identification up to a diffeomorphism is good enough. Scientifically, of course, we might want to know what the state variables actually are; to do this we'll need more than just passive observation of time series.

References

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