

Chapter 11

Markov Examples

Section 11.1 finds the transition kernels for the Wiener process, as an example of how to manipulate such things.

Section 11.2 looks at the evolution of densities under the action of the logistic map; this shows how deterministic dynamical systems can be brought under the sway of the theory we've developed for Markov processes.

11.1 Transition Kernels for the Wiener Process

We have previously defined the Wiener process (Examples 38 and 78) as the real-valued process on \mathbb{R}^+ with the following properties:

1. $W(0) = 0$;
2. For any three times $t_1 \leq t_2 \leq t_3$, $W(t_3) - W(t_2) \perp\!\!\!\perp W(t_2) - W(t_1)$ (independent increments);
3. For any two times $t_1 \leq t_2$, $W(t_2) - W(t_1) \sim \mathcal{N}(0, t_2 - t_1)$ (Gaussian increments);
4. Continuous sample paths (in the sense of Definition 72).

Here we will use the Gaussian increment property to construct a transition kernel, and then use the independent increment property to show that these kernels satisfy the Chapman-Kolmogorov equation, and hence that there exist *Markov* processes with the desired finite-dimensional distributions.

First, notice that the Gaussian increments property gives us the transition probabilities:

$$\mathbb{P}(W(t_2) \in B | W(t_1) = w_1) = \mathbb{P}(W(t_2) - W(t_1) \in B - w_1) \quad (11.1)$$

$$= \int_{B-w_1} du \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{u^2}{2(t_2-t_1)}} \quad (11.2)$$

$$= \int_B dw_2 \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(w_2-w_1)^2}{2(t_2-t_1)}} \quad (11.3)$$

$$\equiv \mu_{t_1, t_2}(w_1, B) \quad (11.4)$$

To show that $W(t)$ is a Markov process, we must show that, for any three times $t_1 \leq t_2 \leq t_3$, $\mu_{t_1, t_2} \mu_{t_2, t_3} = \mu_{t_1, t_3}$.

Notice that $W(t_3) - W(t_1) = (W(t_3) - W(t_2)) + (W(t_2) - W(t_1))$. Because increments are independent, then, $W(t_3) - W(t_1)$ is the sum of two independent random variables, $W(t_3) - W(t_2)$ and $W(t_2) - W(t_1)$. The distribution of $W(t_3) - W(t_1)$ is then the convolution of distributions of $W(t_3) - W(t_2)$ and $W(t_2) - W(t_1)$. Those are $\mathcal{N}(0, t_3 - t_2)$ and $\mathcal{N}(0, t_2 - t_1)$ respectively. The convolution of two Gaussian distributions is a third Gaussian, summing their parameters, so according to this argument, we must have $W(t_3) - W(t_1) \sim \mathcal{N}(0, t_3 - t_1)$. But this is precisely what we should have, by the Gaussian-increments property. Since the trick we used above to get the transition kernel from the increment distribution can be applied again, we conclude that $\mu_{t_1, t_2} \mu_{t_2, t_3} = \mu_{t_1, t_3}$ and the Chapman-Kolmogorov property is satisfied; therefore (Theorem 103), $W(t)$ is a Markov process (with respect to its natural filtration).

To see that $W(t)$ has, or can be made to have, continuous sample paths, invoke Theorem 94.

11.2 Probability Densities in the Logistic Map

Let's revisit the first part of Exercise 5.3, from the point of view of what we now know about Markov processes. The exercise asks us to show that the density $\frac{1}{\pi\sqrt{x(1-x)}}$ is invariant under the action of the logistic map with $a = 4$.

Let's write the mapping as $F(x) = 4x(1-x)$. Solving a simple quadratic equation gives us the fact that $F^{-1}(x)$ is the set $\{\frac{1}{2}(1 - \sqrt{1-x}), \frac{1}{2}(1 + \sqrt{1-x})\}$. Notice, for later use, that the two solutions add up to 1. Notice also that $F^{-1}([0, x]) = [0, \frac{1}{2}(1 - \sqrt{1-x})] \cup [\frac{1}{2}(1 + \sqrt{1-x}), 1]$. Now we consider $\mathbb{P}(X_{n+1} \leq x)$,

the cumulative distribution function of X_{n+1} .

$$\begin{aligned} \mathbb{P}(X_{n+1} \leq x) &= \mathbb{P}(X_{n+1} \in [0, x]) \end{aligned} \quad (11.5)$$

$$= \mathbb{P}(X_n \in F^{-1}([0, x])) \quad (11.6)$$

$$= \mathbb{P}\left(X_n \in \left[0, \frac{1}{2}(1 - \sqrt{1-x})\right] \cup \left[\frac{1}{2}(1 + \sqrt{1-x}), 1\right]\right) \quad (11.7)$$

$$= \int_0^{\frac{1}{2}(1-\sqrt{1-x})} \rho_n(y) dy + \int_{\frac{1}{2}(1+\sqrt{1-x})}^1 \rho_n(y) dy \quad (11.8)$$

where ρ_n is the density of X_n . So we have an integral equation for the evolution of the density,

$$\int_0^x \rho_{n+1}(y) dy = \int_0^{\frac{1}{2}(1-\sqrt{1-x})} \rho_n(y) dy + \int_{\frac{1}{2}(1+\sqrt{1-x})}^1 \rho_n(y) dy \quad (11.9)$$

This sort of integral equation is complicated to solve directly. Instead, take the derivative of both sides with respect to x ; we can do this through the fundamental theorem of calculus. On the left hand side, this will just give $\rho_{n+1}(x)$, the density we want.

$$\rho_{n+1}(x) \quad (11.10)$$

$$\begin{aligned} &= \frac{d}{dx} \int_0^{\frac{1}{2}(1-\sqrt{1-x})} \rho_n(y) dy + \frac{d}{dx} \int_{\frac{1}{2}(1+\sqrt{1-x})}^1 \rho_n(y) dy \\ &= \rho_n\left(\frac{1}{2}(1 - \sqrt{1-x})\right) \frac{d}{dx} \left(\frac{1}{2}(1 - \sqrt{1-x})\right) \end{aligned} \quad (11.11)$$

$$\begin{aligned} &\quad - \rho_n\left(\frac{1}{2}(1 + \sqrt{1-x})\right) \frac{d}{dx} \left(\frac{1}{2}(1 + \sqrt{1-x})\right) \\ &= \frac{1}{4\sqrt{1-x}} \left(\rho_n\left(\frac{1}{2}(1 - \sqrt{1-x})\right) + \rho_n\left(\frac{1}{2}(1 + \sqrt{1-x})\right) \right) \end{aligned} \quad (11.12)$$

Notice that this defines a linear operator taking densities to densities. (You should verify the linearity.) In fact, this is a Markov operator, by the terms of Definition 113. Markov operators of this sort, derived from deterministic maps, are called *Perron-Frobenius* or *Frobenius-Perron* operators, and accordingly denoted by P . Thus an invariant density is a ρ^* such that $\rho^* = P\rho^*$. All the

problem asks us to do is to verify that $\frac{1}{\pi\sqrt{x(1-x)}}$ is such a solution.

$$\rho^* \left(\frac{1}{2} (1 - \sqrt{1-x}) \right) \quad (11.13)$$

$$= \frac{1}{\pi} \left(\frac{1}{2} (1 - \sqrt{1-x}) \left(1 - \left(\frac{1}{2} (1 - \sqrt{1-x}) \right) \right) \right)^{-1/2}$$

$$= \frac{1}{\pi} \left(\frac{1}{2} (1 - \sqrt{1-x}) \frac{1}{2} (1 + \sqrt{1-x}) \right)^{-1/2} \quad (11.14)$$

$$= \frac{2}{\pi\sqrt{x}} \quad (11.15)$$

Since $\rho^*(x) = \rho^*(1-x)$, it follows that

$$P\rho^* = 2 \frac{1}{4\sqrt{1-x}} \rho^* \left(\frac{1}{2} (1 - \sqrt{1-x}) \right) \quad (11.16)$$

$$= \frac{1}{\pi\sqrt{x(1-x)}} \quad (11.17)$$

$$= \rho^* \quad (11.18)$$

as desired.

By Lemma 117, for any distribution ρ , $\|P^n\rho - P^n\rho^*\|$ is a non-increasing function of n . However, $P^n\rho^* = \rho^*$, so the iterates of *any* distribution, under the map, approach the invariant distribution monotonically. It would be very handy if we could show that any initial distribution ρ eventually converged on ρ^* , i.e. that $\|P^n\rho - \rho^*\| \rightarrow 0$. When we come to ergodic theory, we will see conditions under which such distributional convergence holds, as it does for the logistic map, and learn how such convergence in distribution is connected to both pathwise convergence properties, and to the decay of correlations.

11.3 Exercises

Exercise 11.1 (Brownian Motion with Constant Drift) Consider a process $X(t)$ which, like the Wiener process, has $X(0) = 0$ and independent increments, but where $X(t_2) - X(t_1) \sim \mathcal{N}(a(t_2 - t_1), \sigma^2(t_2 - t_1))$. a is called the drift rate and σ^2 the diffusion constant. Show that $X(t)$ is a Markov process, following the argument for the standard Wiener process ($a = 0$, $\sigma^2 = 1$) above. Do such processes have continuous modifications for all (finite) choices of a and σ^2 ? If so, prove it; if not, give at least one counter-example.

Exercise 11.2 (Perron-Frobenius Operators) Verify that P defined in the section on the logistic map above is a Markov operator.