Chapter 12

Generators of Markov Processes

This lecture is concerned with the infinitessimal generator of a Markov process, and the sense in which we are able to write the evolution operators of a homogeneous Markov process as exponentials of their generator.

Take our favorite continuous-time homogeneous Markov process, and consider its semi-group of time-evolution operators K_t. They obey the relationship $K_{t+s} = K_t K_s$. That is, multiplication of the operators corresponds to addition of their parameters, and vice versa. This is reminiscent of the exponential functions on the reals, where, for any $k \in \mathbb{R}$, $k^{(t+s)} = k^t k^s$. In the discrete-parameter case, in fact, $K_t = (K_1)^t$, where integer powers of operators are defined in the obvious way, through iterated composition, i.e., $K^2 f = K \circ (Kf)$. It would be nice if we could extend this analogy to continuous-parameter Markov processes. One approach which suggests itself is to notice that, for any k, there's another real number g such that $k^t = e^{tg}$, and that e^{tg} has a nice representation involving integer powers of g:

$$e^{tg} = \sum_{i=0}^{\infty} \frac{(tg)^i}{i!}$$

The strategy this suggests is to look for some other operator G such that

$$K_t = e^{tG} \equiv \sum_{i=0}^{\infty} \frac{t^i G^i}{i!}$$

Such an operator G is called the *generator* of the process, and the purpose of this section is to work out the conditions under which this analogy can be carried through.

In the exponential function case, we notice that g can be extracted by taking the derivative at zero: $\frac{d}{dt} e^{tg}\big|_{t=0} = g$. This suggests the following definition.

Definition 119 (Infinitessimal Generator) *Let K_t be a continuous-parameter semi-group of homogeneous Markov operators. Say that a function $f \in L_1$ belongs to* $\mathrm{Dom}(G)$ *if the limit*

$$\lim_{h \downarrow 0} \frac{K_h f - K_0 f}{h} \equiv Gf \tag{12.1}$$

exists in an L_1-norm sense, i.e., there exists some element of L_1, which we shall call Gf, such that

$$\lim_{h \downarrow 0} \left\| \frac{K_h f - K_0 f}{h} - Gf \right\| = 0 \tag{12.2}$$

The operator G defined through Eq. 12.1 is called the infinitessimal generator of the semi-group K_t.

Lemma 120 *G is a linear operator.*

Proof: Exercise 12.1. □

Lemma 121 *If μ is an invariant distribution of the semi-group K_t, then, $\forall f \in$ $\mathrm{Dom}(G)$, $\mu G f = 0$.*

Proof: Since μ is invariant, $\mu K_t = \mu$ for all t, hence $\mu K_h f = \mu f$ for all $h \geq 0$ and all f. Since taking expectations with respect to a measure is a linear operator, $\mu(K_h f - f) = 0$, and obviously then $\mu G f = 0$. □

Remark: The converse statement, that if $\mu G f = 0$ for all f, then μ is an invariant measure, requires extra conditions.

You will usually see the definition of the generator written with f instead of $K_0 f$, but I chose this way of doing it to emphasize that G is, basically, the derivative at zero, that $G = dK/dt|_{t=0}$. Recall, from calculus, that the exponential function can k^t be defined by the fact that $\frac{d}{dt}k^t \propto k^t$ (and e can be defined as the k such that the constant of proportionality is 1). As part of our program, we will want to extend this differential point of view. The next lemma builds towards it, by showing that if $f \in \mathrm{Dom}(G)$, then $K_t f$ is too.

Lemma 122 *If G is the generator of the semi-group K_t, and f is in the domain of G, then K_t and G commute, for all t:*

$$K_t G f = \lim_{t' \to t} \frac{K_{t'} f - K_t f}{t' - t} \tag{12.3}$$

$$= G K_t f \tag{12.4}$$

Proof: Exercise 12.2. □

Definition 123 (Time Derivative in Function Space) *For every $t \in T$, let $u(t, x)$ be a function in L_1. When the limit*

$$u'(t_0, x) = \lim_{t \to t_0} \frac{u(t, x) - u(t_0, x)}{t - t_0} \tag{12.5}$$

exists in the L_1 sense, then we say that $u'(t_0)$ is the time derivative *or* strong derivative *of $u(t)$ at t_0.*

Lemma 124 *Let K_t be a homogeneous semi-group of Markov operators with generator G. Let $u(t) = K_t f$ for some $f \in \text{Dom}(G)$. Then $u(t)$ is differentiable at $t = 0$, and its derivative there is Gf.*

PROOF: Obvious from the definitions. □

Theorem 125 *Let K_t be a homogeneous semi-group of Markov operators with generator G, and let $u(t, x) = (K_t f)(x)$, for fixed $f \in \text{Dom}(G)$. Then $u'(t)$ exists for all t, and is equal to $Gu(t)$.*

PROOF: Since $f \in \text{Dom}(G)$, $K_t G f$ exists, but then, by Lemma 122, $K_t G f = G K_t f = Gu(t)$, so $u(t) \in \text{Dom}(G)$ for all t. Now let's consider the time derivative of $u(t)$ at some arbitrary t_0, working from above:

$$\frac{(u(t) - u(t_0)}{t - t_0} = \frac{K_{t-t_0} u(t_0) - u(t_0)}{t - t_0} \tag{12.6}$$

$$= \frac{K_h u_(t_0) - u(t_0)}{h} \tag{12.7}$$

Taking the limit as $h \downarrow 0$, we get that $u'(t_0) = Gu(t_0)$, which exists, because $u(t_0) \in \text{Dom}(G)$. □

Corollary 126 (Initial Value Problems in Function Space) *$u(t) = K_t f$, $f \in \text{Dom}(G)$, solves the initial value problem $u(0) = f$, $u'(t) = Gu(t)$.*

PROOF: Immediate from the theorem. □

Remark: Such initial value problems are sometimes called *Cauchy problems*, especially when G takes the form of a differential operator.

We are now almost ready to state the sense in which K_t is the result of exponentiating G. This is given by the remarkable Hille-Yosida theorem, which in turn involves a family of operators related to the time-evolution operators, the "resolvents", again built by analogy to the exponential functions. Notice that, for any positive constant λ,

$$\int_{t=0}^{\infty} e^{-\lambda t} e^{tg} dt = \frac{1}{\lambda - g} \tag{12.8}$$

from which we could recover g. The left-hand side is just the Laplace transform of e^{tg}.

Definition 127 (Continuous Functions Vanishing at Infinity) *Let Ξ be a locally compact and separable metric space. The class of functions C_0 will consist of functions $f : \Xi \mapsto R$ which are continuous and for which $x \to \infty$ implies $f(x) \to 0$.*

Definition 128 (Resolvents) *Given a continuous-parameter time-homogeneous Markov semi-group K_t, for each $\lambda > 0$, the* resolvent operator *or* resolvent *R_λ is the "Laplace transform" of K_t: for every $f \in C_0$,*

$$(R_\lambda f)(x) \equiv \int_{t=0}^{\infty} e^{-\lambda t}(K_t f)(x)dt \tag{12.9}$$

Remark 1: The name "resolvent", like some of the other ideas an terminology of Markov operators, comes from the theory of integral equations; invariant densities (when they exist) are solutions of homogeneous linear Fredholm integral equations of the second kind. Rather than pursue this analogy, or even explain what that means, I will refer you to the classic treatment of integral equations by Courant and Hilbert (1953, ch. 3), which everyone else seems to follow *very closely*.

Remark 2: When the function f is a value (loss, benefit, utility, ...) function, $(K_t f)(x)$ is the expected value at time t when starting the process in state x. $(R_\lambda f)(x)$ can be thought of as the *net present expected value* when starting at x and applying a discount rate λ.

Definition 129 (Yosida Approximation of Operators) *The* Yosida approximation *to a semi-group K_t with generator G is given by*

$$K_t^\lambda \equiv e^{tG^\lambda} \tag{12.10}$$

$$G^\lambda \equiv \lambda G R_\lambda = \lambda(\lambda R_\lambda - I) \tag{12.11}$$

The domain of G^λ contains all C_0 functions, not just those in Dom(G).

Theorem 130 (Hille-Yosida Theorem) *Let G be a linear operator on some linear subspace $\mathcal{D}$ of L_1. G is the generator of a continuous semi-group of contractions K_t if and only if*

1. *$\mathcal{D}$ is dense in L_1;*
2. *For every $f \in L_1$ and $\lambda > 0$, there exists a unique $g \in \mathcal{D}$ such that $\lambda g - Gg = f$;*
3. *For every $g \in \mathcal{D}$ and positive λ, $\|\lambda g - Gg\| \geq \lambda \|g\|$.*

Under these conditions, the resolvents of K_t are given by $R_\lambda = (\lambda - G)^{-1}$, and K_t is the limit of the Yosida approximations as $\lambda \to \infty$:

$$K_t f = \lim_{\lambda \to \infty} K_t^\lambda f, \ \forall f \in L_1 \tag{12.12}$$

Proof: See Kallenberg, Theorem 19.11. □

12.1 Exercises

Exercise 12.1 *Prove Lemma 120.*

Exercise 12.2 *Prove Lemma 122.*

a *Prove Equation 12.3, restricted to* $t' \downarrow t$ *instead of* $t' \to t$. *Hint: Write* T_t *in terms of an integral over the corresponding transition kernel, and find a reason to exchange integration and limits.*

b *Show that the limit as* $t' \uparrow t$ *also exists, and is equal to the limit from above. Hint: Re-write the quotient inside the limit so it only involves positive time-differences.*

c *Prove Equation 12.4.*