Solution to Homework #1, 36-754

27 January 2006

Exercise 1.1 (The product σ-field answers countable questions)

> Let $\mathcal{D} = \bigcup_S \mathcal{X}^S$, where the union ranges over all countable subsets S of the index set T. For any event $D \in \mathcal{D}$, whether or not a sample path $x \in D$ depends on the value of x_t at only a countable number of indices t.
>
> (a) Show that $\mathcal{D}$ is a σ-field.
>
> (b) Show that if $A \in \mathcal{X}^T$, then $A \in \mathcal{X}^S$ for some countable subset S of T.

Cf. the proof of Theorem 29 in the notes.

(a): We must show that (i) $\Xi^T \in \mathcal{D}$, (ii) $A \in \mathcal{D} \Rightarrow \Xi^T \setminus A \in \mathcal{D}$ and (iii) $A_n \in \mathcal{D} \Rightarrow \bigcup_n A_n \in \mathcal{D}$ for any countable collection of sets A_n.

(i): Pick $S = \{t\}$ for any $t \in T$, and take the base set to be Ξ, i.e, the base set is $\left\{x \in \Xi^T :\ x_t \in \Xi_t\right\}$. Clearly, this set is Ξ^T.

(ii): Fix S. Then for any $A \in \mathcal{X}^S$,

$$\begin{aligned} \Xi^T \setminus A &= \Xi^T \setminus \left(A \times \prod_{t \in T \setminus S} \Xi_t \right) \\ &= \left(\Xi^S \setminus A\right) \times \prod_{t \in T \setminus S} \Xi_t \end{aligned}$$

which is in $\mathcal{X}^S$.

(iii): Take any countable collection of sets $A_n \in \mathcal{D}$. For each such set, there is a corresponding finite set of indices, S_n, for which $A_n \in \mathcal{X}^{S_n}$. Let $S = \bigcup_n S_n$.

Because this is a countable union of denumerable sets, S is itself countable. Now

$$\begin{aligned}
\bigcup_n A_n &= \bigcup_n \left(A_n \times \prod_{t \in T \setminus S_n} \Xi_t \right) \\
&= \bigcup_n \left(A_n \times \prod_{t \in S \setminus S_n} \Xi_t \times \prod_{t \in T \setminus S} \Xi_t \right) \\
&= \left(\bigcup_n A_n \times \prod_{t \in S \setminus S_n} \Xi_t \right) \times \prod_{t \in T \setminus S} \Xi_t
\end{aligned}$$

which is clearly $\in \mathcal{X}^S$. Hence $\mathcal{D}$ is closed under countable unions. □

(b): $\mathcal{X}^T$ is, by definition, the smallest σ-field containing all the finite cylinders. Since every finite cylinder is in $\mathcal{D}$, clearly $\mathcal{X}^T \subseteq \mathcal{D}$. But, by definition, if $A \in \mathcal{D}$, then $A \in \mathcal{X}^S$ for some countable S. □

Source: Billingsley, *Probability and Measure*, third edition, Theorem 36.3 (ii), pp. 492–493.

Exercise 3.1 (Łomnick-Ulam Theorem on infinite product measures)

> Let T be an uncountable index set, and $(\Xi_t, \mathcal{X}_t, \mu_t)$ a collection of probability spaces. Show that there exist independent random variables X_t in Ξ_t with distributions μ_t. *Hint:* use the Ionescu Tulcea theorem on countable subsets of T, and then imitate the proof of the Kolmogorov extension theorem.

Pick any countable collection of indices $J \subset T$. Arrange them in any order; there is a 1-1 correspondence between natural numbers and index values. For each $n > 1$, let κ_n be a kernel from $\prod_{i=1}^{n-1} \Xi_i$ to Ξ_n, which always gives the measure μ_n. (For $n = 1$, set κ_1 to be a kernel from the empty set to Ξ_1 which always gives μ_1.) Then the Ionescu Tulcea Theorem (33) gives us a measure μ_J on $\Xi_J, \mathcal{X}_J$. Moreover, this measure does not depend on the ordering we chose of the indices in J: if we had chosen a different one, we would still get the same finite-dimensional distributions, and consequently (Theorem 23) the same infinite-dimensional distribution.

Now proceed exactly as in the proof of the Kolmogorov Extension Theorem (29), defining a set function on the countable cylinders, and showing that it is countably additive. (You should go through all the steps!) □

Source: Kallenberg, Corollary 6.18, p. 117.