Solution to Homework #2, 36-754

7 February 2006

Exercise 5.3 (The Logistic Map as a Measure-Preserving Transformation)

> The logistic map with $a = 4$ is a measure-preserving transformation, and the measure it preserves has the density $1/\pi\sqrt{x(1-x)}$ (on the unit interval.
>
> 1. Verify that this density is invariant under the action of the logistic map.
> 2. Simulate the logistic map with *uniformly* distributed X_0. What happens to the density of X_t as $t \to \infty$?

a

There are a number of ways to do this; here is one which will feed into later material. It does not use the fact that the invariant density is the $\beta(1/2, 1/2)$ distribution, or that it has a closed-form integral,

$$\int_0^x \frac{dy}{\pi\sqrt{y(1-y)}} = \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

though both of these are fine facts to have handy.

Let's write the mapping as $F(x) = 4x(1-x)$. Solving a simple quadratic equation gives us the fact that $F^{-1}(x)$ is the set $\left\{\frac{1}{2}\left(1-\sqrt{1-x}\right), \frac{1}{2}\left(1+\sqrt{1-x}\right)\right\}$. Notice, for later use, that the two solutions add up to 1. Notice also that $F^{-1}([0,x]) = \left[0, \frac{1}{2}\left(1-\sqrt{1-x}\right)\right] \cup \left[\frac{1}{2}\left(1+\sqrt{1-x}\right), 1\right]$. Now we consider $\mathbb{P}(X_{n+1} \leq x)$, the cumulative distribution function of X_{n+1}.

$$\begin{aligned}
\mathbb{P}(X_{n+1} \leq x) &= \mathbb{P}(X_{n+1} \in [0,x]) \\
&= \mathbb{P}\left(X_n \in F^{-1}([0,x])\right) \\
&= \mathbb{P}\left(X_n \in \left[0, \frac{1}{2}\left(1-\sqrt{1-x}\right)\right] \cup \left[\frac{1}{2}\left(1+\sqrt{1-x}\right), 1\right]\right) \\
&= \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} \rho_n(y)\,dy + \int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^1 \rho_n(y)\,dy
\end{aligned}$$

where ρ_n is the density of X_n. So we have an integral equation for the evolution of the density,

$$\int_0^x \rho_{n+1}(y)\,dy = \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} \rho_n(y)\,dy + \int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^1 \rho_n(y)\,dy$$

This sort of integral equation is complicated to solve directly. Instead, take the derivative of both sides with respect to x; we can do this through the fundamental theorem of calculus. On the left hand side, this will just give $\rho_{n+1}(x)$, the density we want.

$$\begin{aligned}
\rho_{n+1}(x) &= \frac{d}{dx}\int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} \rho_n(y)\,dy + \frac{d}{dx}\int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^1 \rho_n(y)\,dy \\
&= \rho_n\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)\frac{d}{dx}\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right) - \rho_n\left(\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\frac{d}{dx}\left(\frac{1}{2}\left(1+\sqrt{1-x}\right)\right) \\
&= \frac{1}{4\sqrt{1-x}}\left(\rho_n\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right) + \rho_n\left(\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\right)
\end{aligned}$$

Notice that this defines a linear operator taking densities to densities. (You should verify the linearity.) In fact, this is (see Section 10.3 in the notes, especially Definition 113) a Markov operator, as you can verify. Markov operators of this sort, derived from deterministic maps, are called *Perron-Frobenius* or *Frobenius-Perron* operators, and accordingly denoted by P. Thus an invariant density is a ρ such that $\rho = P\rho$. All the problem asks us to do is to verify that $\frac{1}{\pi\sqrt{x(1-x)}}$ is such a solution.

$$\begin{aligned}
\rho\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right) &= \frac{1}{\pi}\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\left(1-\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)\right)\right)^{-1/2} \\
&= \frac{1}{\pi}\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)^{-1/2} \\
&= \frac{2}{\pi\sqrt{x}}
\end{aligned}$$

Since $\rho(x) = \rho(1-x)$, it follows that

$$\begin{aligned}
P\rho &= 2\frac{1}{4\sqrt{1-x}}\rho\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right) \\
&= \frac{1}{\pi\sqrt{x(1-x)}} \\
&= \rho
\end{aligned}$$

as desired.

Source: This is traditional in the study of deterministic measure-preserving systems. See, for example, Lasota and Mackey.

b

Any reasonable language is suitable for this part of the problem. I used R, just to make graphing things easier later.

```
ulam_map <- function(x) {
        # written in vectorized form
        4*x*(1-x)
}

iterate_map <- function(x,map,n) {
        for (i in 1:n) {
                x <- map(x);
        }
        x
}

invariant_density <- function(x) {
        1/(pi*sqrt(x*(1-x)))
}
```

Then, I ran the following:

```
> x <- runif(10000)
> x <- iterate_map(x,ulam_map,1)
> x <- iterate_map(x,ulam_map,4)
> x <- iterate_map(x,ulam_map,5)
> x <- iterate_map(x,ulam_map,90)
```

and, in between,

```
> hist(x,probability=TRUE,main="Histogram of initial density")
> plot(invariant_density,add=TRUE,lty=2)
```

to get a graphical comparison between the distribution of points in the simulation and the invariant density (next page).

To get a more quanitative comparison, I used the Kolmogorov-Smirnov test to compare the empirical distribution of the simulated points with the invariant distribution and got (with a particular initial random sample) the following p-values:

n	p
0	$< 2.2 \cdot 10^{-16}$
1	$< 2.2 \cdot 10^{-16}$
2	$< 2.2 \cdot 10^{-16}$
3	0.1067
4	0.3343
5	0.8182

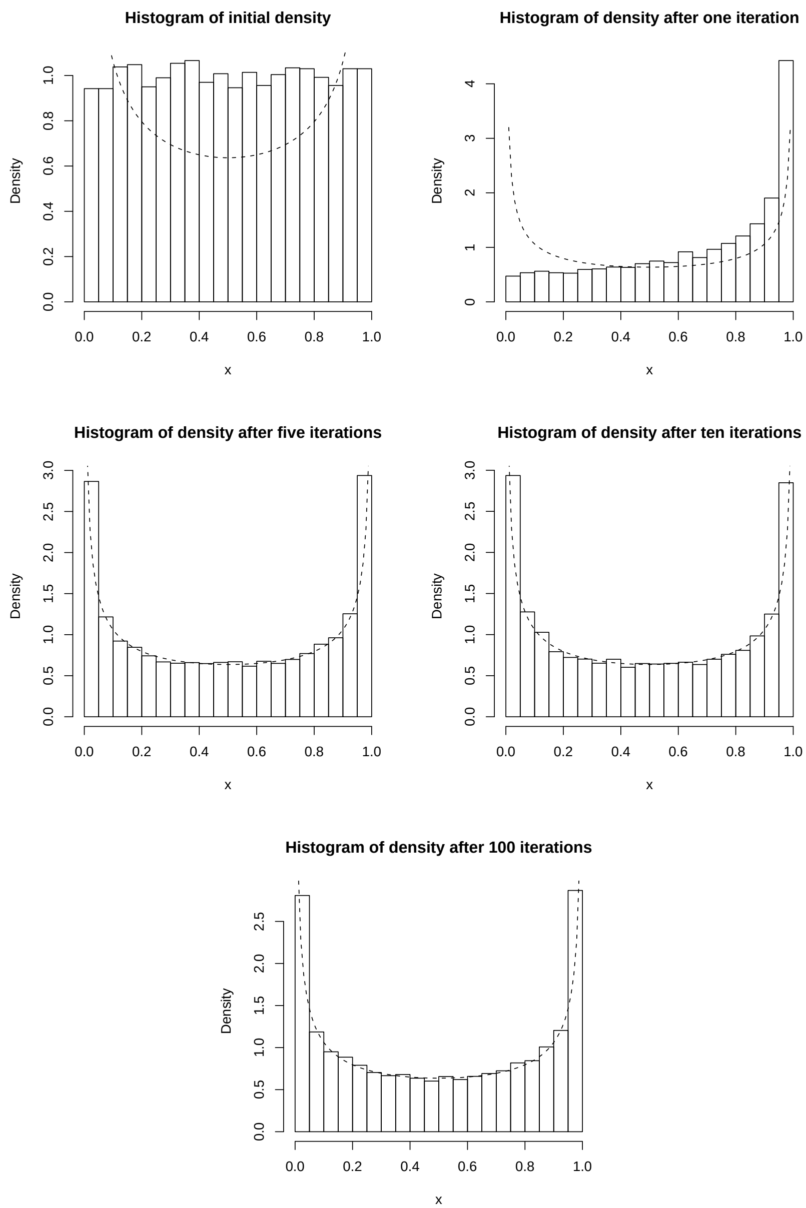

Figure 1: Evolution of the distribution of points in the simulation of the logistic map, starting from a uniform distribution, towards the invariant distribution, shown by the dotted curve for comparison. Note how, even though the initial distribution is symmetric around 1/2, and the map, too, is symmetric around 1/2, after one iteration the distribution is *not* symmetric but peaked around 1. This shows the danger of relying on naive symmetry arguments.

which generally makes the point: after a few iterations, the difference between the simulated distribution and the invariant one is basically the same as what we'd expect by sampling fluctuations.

(We will see later how to connect the KS test to theorems about the convergence of stochastic processes.)

6.1 (Weakly Optional Times and Right-Continuous Filtrations)

> Show that a random time τ is weakly $\mathcal{F}$-optional iff it is $\mathcal{F}^+$-optional.

Recall that τ is weakly $\mathcal{F}$-optional when, for all $t \in T$,

$$\{\omega \in \Omega : \ \tau(\omega) < t\} \in \mathcal{F}_t$$

and $\mathcal{F}$-optional when, for all $t \in T$,

$$\{\omega \in \Omega : \ \tau(\omega) \leq t\} \in \mathcal{F}_t$$

Note: throughout, assume that T possesses at least one countable dense subset $\mathcal{D}$.

"If":

$$\{\omega \in \Omega : \ \tau(\omega) < t\} = \bigcup_{r \in \mathcal{D}: \ r < t} \{\omega \in \Omega : \ \tau(\omega) \leq r\}$$

since $\mathcal{D}$ is countable and dense. Since τ is $\mathcal{F}^+$-optional, each set in the union on the right-hand side is in $\mathcal{F}_r^+ = \bigcap_{s>r} \mathcal{F}_s$. Therefore, every set on the right-hand side is in $\mathcal{F}_t$. Since a countable union of sets in $\mathcal{F}_t$ is itself in $\mathcal{F}_t$, it follows that $\{\omega \in \Omega : \ \tau(\omega) < t\} \in \mathcal{F}_t$, and τ is weakly $\mathcal{F}$-optional.

"Only if": Again, re-write the kind of event we want (an optional time in the strong sense) in terms of the kinds of events we have (weakly optional times).

$$\begin{aligned} \{\omega \in \Omega : \ \tau(\omega) \leq t\} &= \bigcap_{r \in \mathcal{D}: \ r > t} \{\omega \in \Omega : \ \tau(\omega) < r\} \\ &= \bigcap_{r \in \mathcal{D}: \ t < r < t+h} \{\omega \in \Omega : \ \tau(\omega) < r\} \end{aligned}$$

where the second line holds for any positive h. τ is weakly optional, so $\{\omega \in \Omega : \ \tau < r\} \in \mathcal{F}_r$ for all r, and $\mathcal{F}$ is a filtration, so the intersection on the right-hand side in the second equation is in $\mathcal{F}_{t+h}$. Thus

$$\begin{aligned} \{\omega \in \Omega : \ \tau \leq t\} &\in \mathcal{F}_r, \ \forall r > t \\ \{\omega \in \Omega : \ \tau \leq t\} &\in \bigcap_{r > t} \mathcal{F}_r = \mathcal{F}_t^+ \end{aligned}$$

as was to be shown.

Source: Kallenberg, Lemma 7.2 on p. 121, which proves a slightly stronger statement about the σ-fields generated by optional and weakly optional times.

6.2 (Kac's Theorem for the Logistic Map)

> Numerically check Kac's Theorem for the logistic map with $a = 4$. Pick any interval $I \subset [0, 1]$ you like, but be sure not to make it *too* small.
>
> 1. Generate n initial points in I, according to the invariant measure $\frac{1}{\pi\sqrt{x(1-x)}}$. For each point x_i, find the first t such that $F^t(x_i) \in I$, and take the mean over the sample. What happens to this *space average* as n grows?
> 2. Generate a single point x_0 in I, according to the invariant measure. Iterate it N times. Record the successive times $t_1, t_2, \ldots$ at which $F^t(x_0) \in I$, and find the mean of $t_i - t_{i-1}$ (taking $t_0 = 0$). What happens to this *time average* as N grows?

Let's set $I = [0.6, 0.8]$. $\rho(I) = 0.1407385$, so the mean time between recurrences should be 7.105374 iterations.

First, code to generate points in the interval from the invariant distribution.

```
rinvariant_dist_points_in_interval <- function(n,lower,upper) {
        # Quantile transform method:
        # Pr(X =< x| a =< X =< b)
        # == (F(x) - F(a))/(F(b) - F(a))
        # hence if p ~ U(0,1),
        # F^-1(p*(F(b)-F(a)) + F(a)) = x
        # will have the right distribution.
        F.b <- pbeta(lower,0.5,0.5)
        F.a <- pbeta(upper,0.5,0.5)
        x <- runif(n)
        qbeta(x*(F.b-F.a) + F.a,0.5,0.5)
}
```

and then to calculate the n^{th} return time of a single point to a given interval, assuming the initial point starts in the interval.

```
nth_return_time <- function(x,n,lower,upper) {
        # Assumes x is in the interval in question
        returns <- 0
        t <- 0
        while (returns < n) {
                t <- t+1
                x <- ulam_map(x)
                if ((x <= upper) & (x >= lower)) {
                        returns <- returns+1
                }
        }
        t
}
```

a: Space averages

We can calculate the required space average through the functions above. To get an average over $N = 10$ initial points, for instance, we'd do this:

```
> mean(sapply(rinvariant_dist_points_in_interval(10,0.6,0.8),nth_return_time,1,0.6,0.8))
[1] 14.5
```

Here are some of these averages.

N	$\bar{t}$
10	14.5
100	6.61
1000	6.617
10000	7.1662
100000	7.09655

b: Time averages

The mean of the first n return times is simply t_n/n. So we can pick a single initial point, according to the right distribution, and then use the return time function given above. I got $x_0 = 0.7336846$.

n	$\langle t \rangle$
1	1
2	1
5	13.4
10	9.5
100	7.52
1000	6.986
10000	7.1332
100000	7.12278