# Homework 6

36-467, Fall 2018

Due at 6 pm on Wednesday, 17 October 2018

AGENDA: Familiarization with deterministic dynamical systems.

In all these problems, $i = \sqrt{-1}$.

1. *1D* In this problem, suppose that we are dealing with a *deterministic* sequence of variables, $x_0, x_1, \ldots x_t \ldots$. Suppose that they are generated by the rule[1] that $x_{t+1} = bx_t$.
   (a) (5) Suppose that $|b| < 1$. Show that whatever $x_0$ was, $x_t \rightarrow 0$ as $t \rightarrow \infty$.
   (b) (3) Suppose that $b > 1$. Show that $x_t \rightarrow \infty$ if $x_0 > 0$, and that $x_t \rightarrow -\infty$ if $x_0 < 0$.
   (c) (3) Suppose that $b < -1$. Does $x_t$ have a limit? Does the answer depend on $x_0$?
2. *1D, continued* Now suppose $x_{t+1} = a + bx_t$.
   (a) (5) Show that if $b \neq 1$, then $y_t = x_t - \frac{a}{1-b}$ evolves according to the rule $y_{t+1} = by_t$.
   (b) (5) Find $\lim_{t\rightarrow\infty} x_t$ when $|b| < 1$.
   (c) (3) Find $\lim_{t\rightarrow\infty} x_t$ when $b > 1$.
3. *2D* In this problem, $\vec{x}_t$ is a two-dimensional vector, which you can think of as a $2 \times 1$ matrix; $\mathbf{b}$ is a $2 \times 2$ matrix, with eigenvalues $\lambda_1$ and $\lambda_2$, and eigenvectors $\vec{v}_1$ and $\vec{v}_2$; and that $\vec{x}_{t+1} = \mathbf{b}\vec{x}_t$.
   (a) (5) Suppose that the eigenvectors of $\mathbf{b}$ form a basis. Find an expression for $\vec{x}_t$ in terms of the eigenvalues, the eigenvectors, and $\vec{x}_0$.
   (b) (5) Suppose that $|\lambda_1|, |\lambda_2| < 1$. Find an expression for $\lim_{t\rightarrow\infty} \vec{x}_t$. (*Hint:* The answer should be the same for all $\vec{x}_0$).)
   (c) (5) Suppose that $\lambda_1 > 1$ but $|\lambda_2| < 1$. Describe what happens to $\vec{x}_t$ as $t \rightarrow \infty$. *Hint:* This sort of situation is sometimes called a "saddle point". (Try to avoid just Googling the phrase until after you've worked on the problem though.)

[1] A.k.a. "map" or "recurrence relation" or "evolution equation".

4. *2D, continued* Continue with the notation from the previous problem, but fix $\mathbf{b} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

   (a) (5) Plot the first and second coordinates of $x_t$ versus $t$, for $t \in 1 : 20$. Describe the shapes they make.

   (b) (4) Verify that $\lambda_1 = \frac{1+i}{2}$, $\lambda_2 = \frac{1-i}{2}$, $\vec{v}_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \frac{\sqrt{2}}{2} \begin{bmatrix} i \\ 1 \end{bmatrix}$.

   (c) (5) Let $\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find numbers $c_1$ and $c_2$ such that $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$. *Hint:* $c_1$ and $c_2$ will be complex numbers.

   (d) (5) $x_1$ is defined as $\mathbf{b}\vec{x}_0$, which has only real (not complex) numbers as entries. Explain how this is compatible with your answers to Problems 3a and 4c.

   (e) (5) Explain how the eigenvalues in Problem 4b explain the patterns you found in Problem 4a.

5. 1D plus noise Suppose $X_{t+1} = a + bX_t + \epsilon_t$, where $\epsilon_t$ is a series of uncorrelated random variables with expectation 0 and variance $\tau^2$. The initial random variable $X_0$ has expectation $\mu$ and variance $\sigma^2$.

   (a) (3) Find $\mathbb{E}[X_1]$ in terms of the parameters. If $|b| < 1$, find the value of $\mu$, in terms of the other parameters, for which $\mathbb{E}[X_0] = \mathbb{E}[X_1]$.

   (b) (3) Find $\text{Var}[X_1]$ in terms of the parameters. If $|b| < 1$, find the value of $\sigma^2$, in terms of the other parameters, for which $\text{Var}[X_0] = \text{Var}[X_1]$.

   (c) (3) Suppose that $|b| < 1$ and that $\mu$ and $\sigma^2$ meet the conditions you found in the previous two problems. Find $\text{Cov}[X_0, X_1]$ in terms of the parameters. (Simplify as much as possible.)

   (d) (5) Under the same conditions, find $\mathbb{E}[X_2]$, $\text{Var}[X_2]$, $\text{Cov}[X_1, X_2]$, and $\text{Cov}[X_0, X_2]$.

   (e) (3) Still under the same conditions, find $\mathbb{E}[X_t]$ and $\text{Var}[X_t]$ for arbitrary $t$.

   (f) (5) Still under the same conditions, find an expression for $\text{Cov}[X_t, X_{t+h}]$ which is valid for any $t$ and any $h > 0$.

   (g) (5) Under these conditions, is this a stationary process?

RUBRIC (10): The text is laid out cleanly, with clear divisions between problems and sub-problems. The writing itself is well-organized, free of grammatical and other mechanical errors, and easy to follow. Questions which ask for a plot or table are answered with both the figure itself and the command (or commands) use to make the plot. Plots are carefully labeled, with informative and legible titles, axis labels, and (if called for) sub-titles and legends; they are placed near the text of the corresponding problem. All quantitative and mathematical claims are supported by appropriate derivations, included in the text, or calculations in code. Numerical results are reported to appropriate precision. Code is properly integrated with a tool like R Markdown or knitr, and both the knitted file and the source file are submitted. The code is indented, commented, and uses meaningful names. All code is relevant, without dangling or useless commands. All parts of all problems are answered with coherent sentences, and raw computer code or output are only shown when explicitly asked for.