Homework 6

36-467, Fall 2020

Due at 6 pm on Thursday, 15 October 2020

AGENDA: Familiarization with deterministic dynamical systems.

In all these problems, $i = \sqrt{-1}$.

1. *1D* In this problem, suppose that we are dealing with a *deterministic* sequence of variables, $x_0, x_1, \ldots x_t \ldots$. Suppose that they are generated by the rule[1] that $x_{t+1} = bx_t$.
   (a) (5) Suppose that $|b| < 1$. Show that whatever x_0 was, $x_t \rightarrow 0$ as $t \rightarrow \infty$.
   (b) (3) Suppose that $b > 1$. Show that $x_t \rightarrow \infty$ if $x_0 > 0$, and that $x_t \rightarrow -\infty$ if $x_0 < 0$.
   (c) (3) Suppose that $b < -1$. Does x_t have a limit? Does the answer depend on x_0?
2. *1D, continued* Now suppose $x_{t+1} = a + bx_t$.
   (a) (5) Show that if $b \neq 1$, then $y_t = x_t - \frac{a}{1-b}$ evolves according to the rule $y_{t+1} = by_t$.
   (b) (5) Find $\lim_{t\rightarrow\infty} x_t$ when $|b| < 1$, in terms of x_0, a and b.
   (c) (3) Find $\lim_{t\rightarrow\infty} x_t$ when $b > 1$, in terms of x_0, a and b.
3. *2D* In this problem, $\vec{x}_t$ is a two-dimensional vector, which you can think of as a 2×1 matrix; $\mathbf{b}$ is a 2×2 matrix, with eigenvalues λ_1 and λ_2, and eigenvectors $\vec{v}_1$ and $\vec{v}_2$; and that $\vec{x}_{t+1} = \mathbf{b}\vec{x}_t$.
   (a) (5) Suppose that the eigenvectors of $\mathbf{b}$ are orthogonal and form a basis. Find an expression for $\vec{x}_t$ in terms of the eigenvalues, the eigenvectors, and $\vec{x}_0$.
   (b) (5) Suppose that $|\lambda_1|, |\lambda_2| < 1$. Find an expression for $\lim_{t\rightarrow\infty} \vec{x}_t$. (*Hint:* The answer should be the same for all $\vec{x}_0$).)
   (c) (5) Suppose that $\lambda_1 > 1$ but $|\lambda_2| < 1$. Describe what happens to $\vec{x}_t$ as $t \rightarrow \infty$. *Hint:* This sort of situation is sometimes called a "saddle point". (Try to avoid just Googling the phrase until after you've worked on the problem though.)

[1] A.k.a. "map" or "recurrence relation" or "evolution equation".

4. *2D, continued* Continue with the notation from the previous problem, but fix $\mathbf{b} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, and let $\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   (a) (3) Plot the first and second coordinates of x_t versus t, for $t \in 1:20$. Describe the shapes they make.

   (b) (2) Make a scatter-plot showing the second coordinate of x_t against the first coordinate, with lines connecting successive points. Describe the shapes the trajectory makes.

   (c) (3) Verify that $\lambda_1 = \frac{1+i}{2}$, $\lambda_2 = \frac{1-i}{2}$, $\vec{v}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} i \\ 1 \end{bmatrix}$.

   (d) (5) Find numbers c_1 and c_2 such that $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$. *Hint:* c_1 and c_2 will be complex numbers.

   (e) (5) x_1 is defined as $\mathbf{b}\vec{x}_0$, which has only real (not complex) numbers as entries. Explain how this is compatible with your answers to Problems 3a and 4d.

   (f) (5) Explain how the eigenvalues in Problem 4c explain the patterns you found in Problem 4a.

5. 1D plus noise Suppose $X_{t+1} = a + bX_t + \epsilon_t$, where ϵ_t is a series of uncorrelated random variables with expectation 0 and variance τ^2. The initial random variable X_0 has expectation μ and variance σ^2. ϵ_t is uncorrelated with X_t and all earlier X's.

   (a) (3) Find $\mathbb{E}[X_1]$ in terms of the parameters. Assuming $|b| < 1$, find the value of μ, in terms of the other parameters, for which $\mathbb{E}[X_0] = \mathbb{E}[X_1]$.

   (b) (3) Find $\mathrm{Var}[X_1]$ in terms of the parameters. If $|b| < 1$, find the value of σ^2, in terms of the other parameters, for which $\mathrm{Var}[X_0] = \mathrm{Var}[X_1]$.

   (c) (3) Suppose that $|b| < 1$ and that μ and σ^2 meet the conditions you found in the previous two problems. Find $\mathrm{Cov}[X_0, X_1]$ in terms of the parameters. (Simplify as much as possible.)

   (d) (5) Under the same conditions, find $\mathbb{E}[X_2]$, $\mathrm{Var}[X_2]$, $\mathrm{Cov}[X_1, X_2]$, and $\mathrm{Cov}[X_0, X_2]$.

   (e) (3) Still under the same conditions, find $\mathbb{E}[X_t]$ and $\mathrm{Var}[X_t]$ for arbitrary t.

   (f) (5) Still under the same conditions, find an expression for $\mathrm{Cov}[X_t, X_{t+h}]$ which is valid for any t and any $h > 0$.

   (g) (5) Under these conditions, is this a stationary process?

6. (1) How much time did you spend on this problem set?

Rubric (10): The text is laid out cleanly, with clear divisions between problems and sub-problems. The writing itself is well-organized, free of grammatical and other mechanical errors, and easy to follow. All plots, tables, etc., are generated automatically by code embedded in the R Markdown file. Plots are carefully labeled, with informative and legible titles, axis labels, and (if called for) sub-titles and legends; they are placed near the text of the corresponding problem. All quantitative and mathematical claims are supported by appropriate derivations, included in the text, or calculations in code. Numerical results are reported to appropriate precision. Code is properly integrated with a tool like R Markdown or knitr, and both the knitted file and the source file are submitted. The code is indented, commented, and uses meaningful names. All code is relevant, without dangling or useless commands. All parts of all problems are answered with coherent sentences, and raw computer code or output are only shown when explicitly asked for.