Statistical Computing (36-350)

Lecture 14: Simulation I: Generating Random Variables

Cosma Shalizi

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Agenda

- The basic random-variable commands
- Transforming uniform random variables to other distributions
  - The quantile method
  - The rejection method
- Where the uniform random numbers come from

Required Reading: Matloff, chapter 8

R Cookbook, chapter 8

Optional reading: Chambers, section 6.10

Stochastic Simulation

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All of these require drawing random variables from distributions

Built-in Random Variable Generators

`runif`, `rnorm`, `rbinom`, `rpois`, `rexp`, etc. etc.

First argument is always `n`, number of variables to generate

Subsequent arguments are parameters to distribution, and vary with the distribution

Many Distributions at Once

Parameters are recycled:

```
> rnorm(n=4,mean=c(-1000,1000),sd=1)
[1]  -999.3637  1000.4710 -1000.4449  1000.1040
```

Each of the *n* draws can get its own parameters

sample

```
sample(x, size, replace=FALSE, prob=NULL)
```

draw random sample of `size` points from `x`, optionally with replacement and/or weights

`x` can be anything where `length()` makes sense, basically

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`sample(x)` does a random permutation
If `x` is a single number, treat it like `1:x`

```
> sample(5)
[1] 1 4 3 2 5
```

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}

> (obs.diff <- diff.in.means(cats$Sex))
[1] 2.120553
> null.diffs <- replicate(1000,diff.in.means(sample(cats$Sex)))
> summary(null.diffs)
     Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
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> mean(null.diffs > obs.diff)
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Most basic possible **permutation test**; get much trickier

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Exercise: write an improved `resample` that works on vectors or data frames

Why Resample the Data?

```
diff.in.means2 <- function(df) {
  mean(df[df$Sex=="M","Hwt"]) - mean(df[df$Sex=="F","Hwt"])
}

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> sd(resample.diffs)
[1] 0.316297
> quantile(resample.diffs,probs=c(0.025,0.975))
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Bootstrap standard error and confidence limits

Again, gets much more sophisticated

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```
ifelse(runif(n) =< p, 1, 0)
```

or just

```
rbinom(n,size=1,prob=p)
```

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min(which(u < cumsum(p)))
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(needs some thought to vectorize)

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```
rmultinoulli <- function(n,prob) {
  return(sample(1:length(prob),replace=TRUE,size=n,prob=prob))
}
```

(`rmultinom` gives counts, not a sequence)

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Given: uniform random variable U, CDF F
Claim: $X = F^{-1}(U)$ is a random variable with CDF F

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Given: uniform random variable U, CDF F
Claim: $X = F^{-1}(U)$ is a random variable with CDF F
Proof:

$$\mathbb{P}(X \leq a) = \mathbb{P}(F^{-1}(U) \leq a) = \mathbb{P}(U \leq F(a)) = F(a)$$

F^{-1} is the quantile function
$\therefore$ if we can generate uniforms and we can calculate quantiles, we can generate non-uniforms

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Tedious do this iteratively

No next value for continuous random variables

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Quantiles solve both difficulties

Quantile functions often don't have closed form, and don't have nice numerical solutions

But we know the probability density function — can we use that?

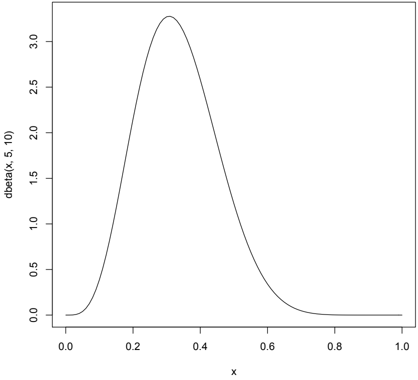
Suppose the pdf f is zero outside an interval $[c, d]$, and $\leq M$ on the interval

Draw the rectangle $[c, d] \times [0, M]$, and the curve f

Area under the curve $= 1$

Area under curve and $x \leq a$ is $F(a)$

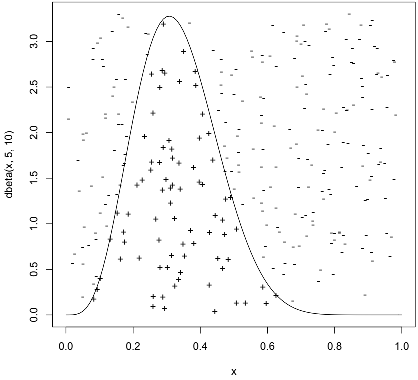
How can we uniformly sample area under the curve?



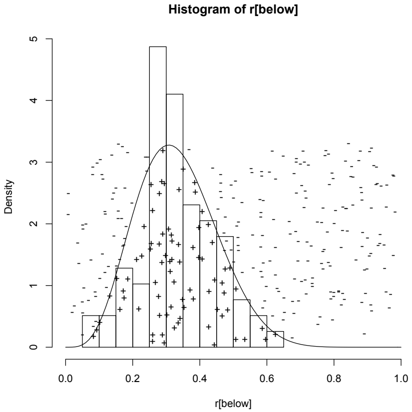

```
M <- 3.3; curve(dbeta(x,5,10),from=0,to=1,ylim=c(0,M))
```

We sample uniformly from the *box*, and take the points under the curve

We sample uniformly from the *box*, and take the points under the curve
$R \sim \mathrm{Unif}(c, d)$
$U \sim \mathrm{Unif}(0, 1)$
If $MU \leq f(R)$ then $X = R$, otherwise try again




```
r <- runif(300,min=0,max=1); u <- runif(300,min=0,max=1)
below <- which(M*u <= dbeta(r,5,10))
points(r[below],M*u[below],pch="+"); points(r[-below],M*u[-below],pch="-")
```




```
hist(r[below],xlim=c(0,1),probability=TRUE); curve(dbeta(x,5,10),add=TRUE)
points(r[below],M*u[below],pch="+"); points(r[-below],M*u[-below],pch="-")
```

If f doesn't go to zero outside $[c, d]$, try to find another density ρ where

- ρ also has unlimited support
- $f(a) \leq M\rho(a)$ everywhere
- we can generate from ρ (say by quantiles)

Then $R \sim \rho$, and accept when $MU\rho(R) \leq f(R)$
(Uniformly distributed on the area under ρ)

Need to make multiple "proposals" R for each X

e.g., generated 300 for figure, only accepted 78

Important for efficiency to keep this ratio small

Ideally: keep the proposal distribution close to the target

Where Do the Uniforms come From?

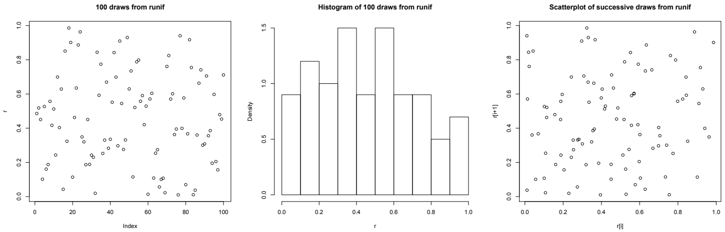
Uniform numbers come from finite algorithms, so really only **pseudo-random**

We want:

- Number of U_i in $[a, b] \subseteq [0, 1]$ is $\propto (b - a)$
- No correlation between successive U_i
- No detectable dependences in larger or longer groupings

Modern pseudo-random generators are now very good at all three

Too involved to go into here, but will show a simpler cousin




```
plot(r,main="100 draws from runif")
plot(hist(r),freq=FALSE,main="Histogram of 100 draws from runif")
plot(r[-100],r[-1],xlab="r[i]",ylab="r[i+1]",
  main="Scatterplot of successive draws from runif")
```

Rotations

Take

$$U_{i+1} = U_i + \alpha \bmod 1$$

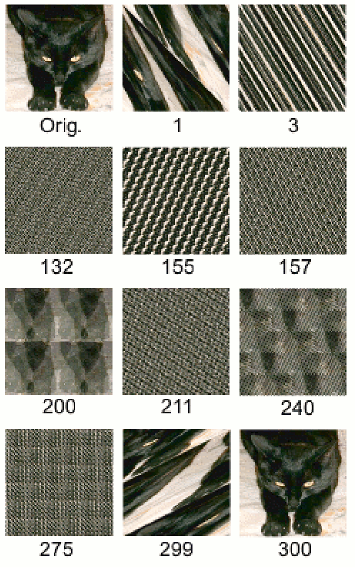
If α is irrational, this never repeats and is uniformly distributed
If α is rational but the denominator is very large, the period is very long, and it is uniform on those points

More Complicated Dynamics

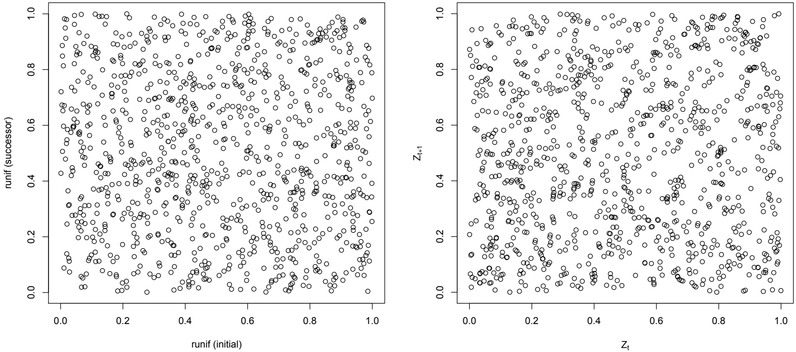
Arnold Cat Map:

$$\begin{aligned} U_{t+1} &= U_t + \phi_t \bmod 1 \\ \phi_{t+1} &= U_t + 2\phi_t \bmod 1 \end{aligned}$$

If we report only U_t, the result is uniformly distributed and hard to predict



Wikipedia, s.v. "Arnold's cat map"



successive values from `runif` vs. Arnold cat map

Similar ideas are built into the random-number generator in R (with more internal dimensions)

Generally: Long periods, rapid divergence of near-by points (unstable), uniform distribution, low correlation

Using the default generator is a very good idea, unless you really know what you are doing

Setting the Seed

The sequence of pseudo-random numbers depends on the initial condition, or **seed**

Stored in `.Random.seed`, a global variable

To reproduce results exactly, set the seed

```
> old.seed <- .Random.seed # Store the seed
> set.seed(20010805)  # Set it to the day I adopted my cat
> runif(2)
[1] 0.1378908 0.7739319
> set.seed(20010805)  # Reset it
> runif(2)
[1] 0.1378908 0.7739319
> .Random.seed <- old.seed  # Restore old seed
```

See Chambers, §6.10, for some subtleties about working with external programs

Summary

- Unstable dynamical systems give us something very like uniform random numbers
- We can transform these into other distributions when we can compute the distribution function
    - The quantile method when we can invert the CDF
    - The rejection method if all we have is the pdf
- The basic R commands encapsulate a lot of this for us