36-703 Homework #1 Solutions Thursday 26 Jan 2006

1

Pentagon Walk Revisited

Following the example from class, we can write

$$S_B(z) = \frac{z^2 - 2z}{(z-1)(z-2\phi)(z-2\widehat{\phi})}$$
$$= \frac{\frac{z^2}{4} - \frac{z}{2}}{(1-z)(1-\frac{z}{2\phi})(1-\frac{z}{2\widehat{\phi}})}.$$

Solving for a partial fractions expansion of the form

$$\frac{u_1}{1-z} + \frac{u_2}{1-\frac{z}{2\phi}} + \frac{u_3}{1-\frac{z}{2\widehat{\phi}}}.$$

Hence,

$$S_B(z) = \frac{1}{5}\left[\frac{u_1}{1-z} + \frac{u_2}{1-\frac{z}{2\phi}} + \frac{u_3}{1-\frac{z}{2\widehat{\phi}}}\right]$$
$$= \frac{1}{5}\left[\frac{u_1}{1-z} + \frac{u_2}{1+\frac{\widehat{\phi}}{2}z} + \frac{u_3}{1+\frac{\phi}{2}z}\right]$$
$$= \frac{1}{5}\sum_{n=0}^{\infty}\left[5u_1 + \left(-\frac{1}{2}\right)^n (5u_2\widehat{\phi}^n + 5u_3\phi^n)\right] z^n.$$

Solving for u_1, u_2, u_3 yields

$$S_B(z) = \frac{1}{5}\sum_{n=0}^{\infty}\left[1 - \left(-\frac{1}{2}\right)^n (\phi^{n+1} + \widehat{\phi}^{n+1})\right] z^n.$$

This gives the right values for smallish n and asymptotes properly to $1/5$ as $n \to \infty$.

2

Double Heads Revisited

Held til next week.

3 Semi-Circle

Describe the Experiment. Choose n points on the unit circle.

State your assumptions. The points are uniformly distributed over the circle and are all chosen independently. If we measure positions on the circle by an angle in the interval $[0, 2\pi)$, counter-clockwise from east, say, then the points are independent Uniform$\langle 0, 2\pi \rangle$ random variables.

Define relevant random variables.

Let A_i be the angular position (from 0 to 2π), measured clockwise from east, of the ith chosen point.

Let S be the indicator that all n points lie within a semi-circle.

Let C_i be the indicator that all n points lie within a semi-circle clockwise from the ith chosen point.

State what you know.

We know that the A_i's are independent and have Uniform$\langle 0, 2\pi \rangle$ distributions.

We also know that there is a relationship between the C_i's and S, which we will clarify below.

State what you want to find.

The desired probability is just $\mathsf{E}S$.

Find it.

Before making a formal argument, let's take a look at the intuitive argument. Suppose you know the position of the ith point. What is the probability that the other $n-1$ points all lie in the $180°$ arc clockwise from this point? Well, because the positions are all independent and uniform on the circle and because the semi-circle is half the circle, the probability is $(1/2)^{n-1}$. Notice that this depends neither on the specific position of the ith point nor on i, both of which make sense by symmetry. When the points all lie within a semi-circle, there can be n possible points that are the counter-clockwise-most, and they all have the same probability of playing this role. Thus, the desired probability is $n(1/2)^{n-1}$.

Now, let's formalize this argument. First, note that if $i \neq j$, then C_i and C_j cannot both be 1 because there can be at most one counter-clockwise-most point when the points all lie within a semi-circle. Moreover, if the points all lie within a semi-circle, then there *must be* a counter-clockwise-most point. Thus, one and only one of the C_i's must equal 1. It follows that

$$S = C_1 + \cdots + C_n,$$

which suggests that we apply the additivity of expected values below. (Put another way, the event that the points all lie within a semi-circle, $\{S = 1\}$, is the *disjoint union* of the events $\{C_i = 1\}$.)

Next, we $\mathsf{E}C_i$ by conditioning on A_i. Suppose A_i were known exactly, then $C_i = 1$ if and only if the other $n-1$ points fall in the semi-circle clockwise from A_i. Each point has probability 1/2 of doing so, and all points are independent; hence,

$$\mathsf{E}(C_i \mid A_i) = \left(\frac{1}{2}\right)^{n-1}.$$

Notice that this depends on neither i nor on A_i. We have by the mighty conditioning identity that

$$\mathsf{E}C_i = \mathsf{E}(\mathsf{E}(C_i \mid A_i)) = \left(\frac{1}{2}\right)^{n-1}.$$

Consequently, by additivity

$$\begin{aligned}
\mathsf{E}S &= \mathsf{E}(C_1 + \cdots + C_n) \\
&= \mathsf{E}C_1 + \cdots + \mathsf{E}C_n \\
&= n\mathsf{E}C_1 \\
&= n\left(\frac{1}{2}\right)^{n-1}.
\end{aligned}$$

This is the desired probability. It is worth checking this in the special case $n = 2$, where it gives the right answer.

4 Mean Free Path

Describe the Experiment.

Another molecule, identical to the others, is inserted at the origin and sent along the y axis at some velocity.

Specify your assumptions.

Assume that the number of molecules within the disk of radius R have a Poisson$\langle\lambda\rangle$ distribution and that the positions of all molecules are independent and uniform over the disk.

Define relevant random variables.

Let N be the number of molecules in the volume (not including the inserted one).

Let D be the distance that the inserted molecule travels up the y axis before hitting another molecule or the edge of the volume (at radius 1).

Let $Z_1, Z_2, \ldots$, be the positions of the molecules in the gass (not including the inserted ones).

State what you know.

We know that N has a Poisson$\langle\lambda\rangle$ distribution and that the Z_is are independent and uniform over the disk. We also know that D takes values between 0 and R.

State what you want to find.

We want to find $\mathsf{E}D$.

Find it.

We will find S_D and then integrate this to get $\mathsf{E}D$.

Let's begin by ignoring the boundary. In this case, the event $\{D > \ell\}$ occurs if and only if there are no other molecules in the region obtained by dragging the circle of radius $2r$ from 0 to ℓ. This is a rectangle of side lengths ℓ and $4r$ capped by two semi-circles of radius $2r$.

Since the positions of the molecules are uniform, each molecule is not in that region with probability

$$1 - \frac{\ell 4r + 4\pi r^2}{\pi R^2} = 1 - 4\frac{r^2}{R^2} - \frac{4}{\pi}\frac{r}{R}\frac{\ell}{R}.$$

If there are n molecules, the probability that no molecules are in this region is this quantity raised to the nth power. We thus need to condition on N.

$$\mathsf{P}\{D > \ell \mid N\} = \left(1 - \frac{4}{\pi}\frac{r}{R}\frac{\ell}{R} - 4\frac{r^2}{R^2}\right)^N.$$

Let $u(\ell) = 1 - \frac{4}{\pi}\frac{r}{R}\frac{\ell}{R} - 4\frac{r^2}{R^2}$ for convenience. Hence,

$$\begin{aligned}
\mathsf{P}\{D > \ell\} &= \mathsf{EP}\{D > \ell \mid N\} \\
&= \mathsf{E}\left(1 - \frac{4}{\pi}\frac{r}{R}\frac{\ell}{R} - 4\frac{r^2}{R^2}\right)^N \\
&= \mathsf{E}u(\ell)^N \\
&= \sum_{k=0}^{\infty} (u(\ell))^k \, e^{-\lambda} \, \frac{\lambda^k}{k!} \\
&= e^{\lambda(u(\ell)-1)}.
\end{aligned}$$

It follows that

$$\mathsf{P}\{D > \ell\} = e^{-4\lambda r^2/R^2} \, e^{-\lambda\frac{4}{\pi}\frac{r}{R}\frac{\ell}{R}}.$$

Note that $D \leq R$ by definition. Integrating $\mathsf{P}\{D > \ell\}$ all the way to the boundary (ignoring edge effects for the moment), yields:

$$\begin{aligned}
\mathsf{E}D &= \int_0^R \mathsf{P}\{D > \ell\} \, d\ell \\
&= e^{-4\lambda r^2/R^2} \int_0^R e^{-\lambda\frac{4}{\pi}\frac{r}{R}\frac{\ell}{R}} \, d\ell \\
&= e^{-4\lambda r^2/R^2} \, R \int_0^1 e^{-\lambda\frac{4}{\pi}\frac{r}{R}\, t} \, dt \\
&= e^{-4\lambda r^2/R^2} \, \frac{\pi R^2}{4r\lambda} \, (1 - e^{-\lambda\frac{4}{\pi}\frac{r}{R}}).
\end{aligned}$$

When $r << R$ so that, say, r^2/R is negligible, we have

$$\mathsf{E}D \approx R - \frac{2r}{\pi}\lambda.$$

But to get a better feel, let's reparametrize $\lambda = \rho\pi R^2$, so that ρ is the expected density of points. Then,

$$\mathsf{E}D = \frac{1}{4r\rho} \, e^{-4\pi r^2 \rho} \, (1 - e^{-4Rr\rho}).$$

This converges to R as $\rho \to 0$ and to 0 as $\rho \to \infty$, and decays roughly exponentially as ρ grows.

Now, to the boundary. This only "kicks in," so to speak, when $\ell \geq R - 2r$. When $r << R$, the contribution to the mean should be small. The simplest approximation might be to replace the one spherical end cap with a square end cap over this range. This gives for $\ell \geq R - 2r$,

$$\mathsf{P}\{D > \ell\} = e^{-2\lambda r^2/R^2} \, e^{-\lambda\frac{2}{\pi}\frac{r}{R}}.$$

Then compute the integral in two stages. For small r, the degree of approximation is very close.

5 — Random Permutations

I should have held this for next week, sorry. Let $i_1, \ldots, i_n$ be any permutation of $1, \ldots, n$. Using the multiplication rule for conditional probabilities:

$$\begin{aligned}
\mathsf{P}\{X_1 = i_1 \text{ and } \cdots \text{ and } X_n = i_n\} \\
&= \mathsf{P}\{X_n = i_n\}\ \mathsf{P}\{X_{n-1} = i_{n-1} \mid X_n = i_n\} \cdots \\
&\qquad \mathsf{P}\{X_1 = i_1 \mid X_2 = i_2 \text{ and } \cdots \text{ and } X_n = i_n\} \\
&= \frac{1}{n}\,\frac{1}{n-1} \cdots \frac{1}{1} \\
&= \frac{1}{n!}.
\end{aligned}$$

We have that

$$\mathsf{P}\{X_j = i_j \mid X_{j+1} = i_{j+1} \text{ and } \cdots \text{ and } X_n = i_n\} = \frac{1}{j}$$

because once $X_{j+1}, \ldots, X_n$ have been determined the array contains $i_{j+1}, \ldots, i_n$ in positions $a_{j+1}, \ldots, a_n$. The algorithm chooses only from the j values in the first j slots, and each of the j possibilities (including i_j) are equally likely.

6 — Rejection Sampling

Describe the Experiment. Generate a sequence of random pairs, Uniform in the rectangle $(a, b) \times (0, m)$. Return the first component of the first "accepted" pair.

Specify your assumptions. All generated pairs are independent; the components X and Y of each pair are independent of each other. In short, we assume that all the Uniform random numbers used in the procedure are independent.

Define relevant random variables

Let X_i and Y_i be the random pair generated on the ith iteration.

Let A be the number of the iteration on which the pair is first accepted.

Let Z be the horizontal coordinate of the accepted pair.

State what you know

We know that the X_i's have a Uniform$\langle a, b\rangle$ distribution.

We know that the Y_i's have a Uniform$\langle 0, m\rangle$ distribution.

We know that the X_i's and Y_i's are all independent and that the pairs (X_i, Y_i) all have the same distribution.

We know that $A = i$ if and only if $Y_i \leq f(X_i)$ and for $1 \leq j < i$ $Y_j > f(X_j)$.

State what you want to find.

We want to find the cdf F_Z. Thus, for each u, we will find $F_Z(u) = \mathsf{P}\{Z \leq u\}$.

Find it.

First, notice that the event $\{A = i\}$ occurs only if the first $i-1$ pairs are rejected and the ith pair accepted. That is,

$$\{A = i\} = \{Y_i \leq f(X_i) \text{ and for } 1 \leq j < i \; Y_j > f(X_j)\}.$$

Second, as suggested in the hints, we condition on A.

$$F_Z(u) = \mathsf{P}\{Z \leq u\} = \sum_{i=1}^{\infty} \mathsf{P}\left(\{Z \leq u\} \mid \{A = i\}\right) \mathsf{P}\{A = i\}.$$

We start with the conditional probability:

$$\begin{aligned}
\mathsf{P}\left(\{Z \leq u\} \mid \{A = i\}\right) &= \mathsf{P}\{X_i \leq u \mid A = i\} \\
&= \mathsf{P}\{X_i \leq u \mid Y_i \leq f(X_i) \text{ and for } 1 \leq j < i \; Y_j > f(X_j)\} \\
&= \mathsf{P}\{X_i \leq u \mid Y_i \leq f(X_i)\} \\
&= \frac{\mathsf{P}\{X_i \leq u \text{ and } Y_i \leq f(X_i)\}}{\mathsf{P}\{Y_i \leq f(X_i)\}} \\
&= \frac{\mathsf{P}\{X_1 \leq u \text{ and } Y_1 \leq f(X_1)\}}{\mathsf{P}\{Y_1 \leq f(X_1)\}} \\
&= \frac{\int_a^u \frac{f(x)}{m} \frac{1}{b-a}\, dx}{\int_a^b \frac{f(x)}{m} \frac{1}{b-a}\, dx} \\
&= \frac{F(u)}{m(b-a)} \cdot \frac{1}{\frac{1}{m\cdot(b-a)}} \\
&= F(u).
\end{aligned}$$

The first equality follows from the meaning of $A = i$. The second equality follows by substitution using the above event equality. The third equality follows from independence. The fourth equality follows from the fact that the (X_j, Y_j) pairs are identically distributed. The fifth equality follows directly by conditioning on X_i in both the numerator and denominator. These can be done more formally by looking at the pictures. We are choosing a random pair (X_1, Y_1) uniformly in the rectangle $(a, b) \times (0, m)$. The area under the pdf f is 1 by definition of a pdf; the area under the pdf to the left of u is $F(u)$; the total area of the rectangle is $m \cdot (b - a)$. Hence, the relative areas are $\mathsf{P}\{Y_1 \leq f(X_1)\} = \frac{1}{m \cdot (b-a)}$ and $\mathsf{P}\{X_1 \leq u \text{ and } Y_1 \leq f(X_1)\} = \frac{F(u)}{m \cdot (b-a)}$. Notice that the conditional distribution of X_i given that $A = i$ is not the same as the unconditional distribution of X_i.

Consequently, we have hat

$$\begin{aligned} F_Z(u) &= \mathsf{P}\{Z \leq u\} \\ &= \sum_{i=1}^{\infty} \mathsf{P}\left(\{Z \leq u\} \mid \{A = i\}\right) \mathsf{P}\{A = i\} \\ &= F(u) \sum_{i=1}^{\infty} \mathsf{P}\{A = i\} \\ &= F(u). \end{aligned}$$

So, it was not necessary to compute $\mathsf{P}\{A = i\}$ at all. And we are done!

If you are curious, notice that using the above event equality.

$$\begin{aligned} \mathsf{P}\{A = i\} &= \mathsf{P}\{Y_i \leq f(X_i) \text{ and for } 1 \leq j < i \; Y_j > f(X_j)\} \\ &= \mathsf{P}\{Y_i \leq f(X_i)\} \cdot \prod_{j=1}^{i-1} \mathsf{P}\{Y_j > f(X_j)\} \\ &= \mathsf{P}\{Y_1 \leq f(X_1)\} \cdot \left(1 - \mathsf{P}\{Y_1 \leq f(X_1)\}\right)^{i-1}. \end{aligned}$$

The last two equalities follow from the fact that the pairs (X_j, Y_j) are independent and identically distributed, respectively. Also, of course, $\mathsf{P}\{Y \leq f(X)\} = 1 - \mathsf{P}\{Y > f(X)\}$. Hence, A has a Geometric$\langle \frac{1}{m \cdot (b-a)} \rangle$ distribution.