36-703 Homework #3 Solutions Thursday 16 Feb 2006

1

LIF-ting

Since we are interested in just $T(z)$, F is the identity function, i.e. $F(x) = x$.
So, $F'(x) = 1$. Let $u = T(z) \Rightarrow G(u) = e^u$. Then the coefficients t_n satisfy

$$[z^n]T(z) = \frac{1}{n}\left[u^{n-1}\right]e^{nu}$$

Generating function of e^{nu} equals

$$\sum_{n\geq 0}\frac{n^n}{n!}u^n$$

It follows that

$$\frac{1}{n}\left[u^{n-1}\right]e^{nu} = \frac{n^{n-1}}{n!}$$

2

Matrix Expansion

(a) In Pentagon example, we've basically witnessed that for each m elements in $\mathbf{p}_n$,

$$\mathbf{p}_{nm} = \mathbf{p}_0 I_{\{n=0\}} + \sum_i \mathsf{P}(X_n = m \mid X_{n-1} = i)\mathsf{P}(X_{n-1} = i)$$

Therefore,

$$\mathbf{p}_n = \mathbf{p}_0 I_{\{n=0\}} + \mathbf{p}_{n-1}\mathbf{P}$$

Plug in the above value of $\mathbf{p}_n$ into the definition of generating function $G(z)$, i.e.

$$G(z) = \sum_n \mathbf{p}_n z^n$$

and the result follows after a little algebra.

(b) This is very similar to the one-dimensional case, $\dfrac{1}{1-pz}$. So one guess would be

$$\sum_{k=0}^{\infty}\mathbf{P}^k z^k$$

In the general case, given an invertible matrix $\mathbf{P}$ and scalar a,

$$(\mathbf{I} \pm a\mathbf{P})^{-1} = \mathbf{I} + \sum_{k=1}^{\infty}(\mp 1)^k \mathbf{P}^k a^k$$

Let $\mathbf{I} = \mathbf{P}^0 z^0$ and substitute a with z. Our guess is confirmed.

(c) It is given by the coefficients of $\mathbf{p}_0(\mathbf{I} - z\mathbf{P})^{-1}$, which is $\mathbf{p}_0\mathbf{P}^n$.

3

Double Roots

In double roots case, we end up with a system of equations that has one equation but two unknowns. One alternative solution involves using the fact that the derivative of a polynomial of order 2 evaluates to zero if its root has multiplicity 2. Using this fact, we can differentiate

$$G(z)(az^2 + bz + c) = zP(z; g_1, g_{m-1})$$

and evaluate it at its root to get a second equation.

If one of the roots is zero, then $c = 0$. So $cg_1 = 0$ and as a result, we only have to solve for g_{m-1}. When both are zero, then $b = c = 0$. The recurrence equations become

$$\begin{aligned} ag_{k-1} &= u_k \\ g_0 &= u_0 \\ g_m &= u_m \end{aligned}$$

Using boundary conditions, we can determine all of the g_k recursively. Therefore, we don't have anything to solve! Not very interesting...

4

Poisson Naturally

(a) First divide $(0, t]$ into n intervals of length t/n.
Next, let $\epsilon_i = 1$ if 1 event occurs on the interval $((i-1)t/n, it/n]$, and 0 otherwise.
So, $S_n = \epsilon_1 + \ldots + \epsilon_n \sim \text{Bin}(n, p)$ where $p = \dfrac{\lambda t}{n} + o(t/n)$.

Using the Poisson approximation of Binomial probabilities,

$$\begin{aligned} \mathsf{P}(S_n = k) &= \frac{e^{-np}(np)^k}{k!} \\ &= \frac{e^{-\lambda t - tz}(\lambda t + tz)^k}{k!} \qquad \text{note } z = \frac{o(t/n)}{t/n} \end{aligned}$$

$$\to \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad \text{as } n \to \infty$$

(b) $\mathsf{P}(S_1 \le s \mid N_t = 1)$

$$\begin{aligned}
&= \mathsf{P}(N_s = 1 \mid N_t = 1) \\
&= \frac{\mathsf{P}(N_s = 1 \cap N_t = 1)}{\mathsf{P}(N_t = 1)} \\
&= \frac{\mathsf{P}(N_t = 1 \mid N_s = 1)\mathsf{P}(N_s = 1)}{\mathsf{P}(N_t = 1)} \\
&= s/t \quad \text{for } s < t
\end{aligned}$$

This is the cdf of Uniform$(0, t)$.

(c) We'll solve the general case in part (d). The argument is essentially the same.

(d) For simplicity, let $s_0 = 0$. For $0 \le s_1 \le s_2 \ldots \le s_n \le t$,
$\mathsf{P}(S_1 \le s_1, \ldots, S_n \le s_n \mid N_t = n)$

$$\begin{aligned}
&= \frac{\mathsf{P}(S_1 \le s_1, \ldots, S_n \le s_n, N_t = n)}{\mathsf{P}(N_t = n)} \\
&= \frac{\mathsf{P}(N_t - N_{s_n} = 0)\mathsf{P}(N_{s_n} - N_{s_{n-1}} = 1) \cdots \mathsf{P}(N_{s_1} - N_{s_0} = 1)}{\mathsf{P}(N_t = n)} \\
&= \frac{\exp\{-\lambda(t - s_n)\} \left(\prod_{j=1}^n \exp\{-\lambda(s_j - s_{j-1})\}\lambda(s_j - s_{j-1})\right)}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} \\
&= \frac{\lambda^n e^{-\lambda t} \prod_{j=1}^n (s_j - s_{j-1})}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} \\
&= \frac{n!}{t^n} \prod_{j=1}^n (s_j - s_{j-1})
\end{aligned}$$

Take derivative with respect to $s_1, \cdots, s_n$ to get the joint pdf.

The resulting conditional joint pdf is $\frac{n!}{t^n}$,for $t > 0$

This is the pdf of the order statistics for a random sample of n Uniform$(0, t)$ random variables.

5

A Poisson Diet Plan

Let $N(t), N_1(t)$ and $N_2(t)$ denote the number of customers entering the building, "Good Eats", and "Eat Good" in $[0,t]$. The claim is $N_1(t)$ is a Poisson process with rate $p\lambda$ and $N_2(t)$ is a Poisson process with rate $q\lambda$. Using law of total probability,

$$\begin{aligned}
\mathsf{P}(N_1(t)=k) &= \sum_{i=0}^{\infty} \mathsf{P}(N_1(t)=k|N(t)=i)\mathsf{P}(N(t)=i) \\
&= \sum_i \binom{i}{k} p^k q^{i-k} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \\
&= \frac{(p\lambda t)^k e^{-\lambda t}}{k!} \sum_i \frac{[(1-p)\lambda t]^{i-k}}{(i-k)!} \\
&= \frac{(p\lambda t)^k e^{-\lambda t}}{k!} e^{(1-p)\lambda t} \\
&= \frac{(p\lambda t)^k e^{-p\lambda t}}{k!}
\end{aligned}$$

We can do the same for $N_2(t)$.

6

Martingales

(a) Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n), \mathcal{G}_n = \sigma(Y_0, \ldots, Y_n)$

$$\begin{aligned}
\mathsf{E}(X_{n+1} \mid \mathcal{F}_n) &= \mathsf{E}[\mathsf{E}(Z \mid \mathcal{G}_{n+1}) \mid \mathcal{F}_n] \\
&= \mathsf{E}(\mathsf{E}[\mathsf{E}(Z \mid \mathcal{G}_{n+1}) \mid \mathcal{G}_n] \mid \mathcal{F}_n) \\
&= \mathsf{E}[\mathsf{E}(Z \mid \mathcal{G}_n) \mid \mathcal{F}_n] \\
&= \mathsf{E}[X_n \mid \mathcal{F}_n] = X_n
\end{aligned}$$

Second equality holds because $\mathcal{F}_n \subset \mathcal{G}_n$. The subset relationship is due to the fact that X_n is a function of the random variables in $\mathcal{G}_n$. Third equality holds by the Mighty Conditioning Identity.

(b)

$$\begin{aligned}
\mathsf{E}(X_{n+1} \mid \mathcal{F}_n) &= \mathsf{E}\left(X_n \frac{f_1(Y_{n+1})}{f_0(Y_{n+1})} \,\middle|\, \mathcal{F}_n \right) \\
&= X_n \mathsf{E}\left(\frac{f_1(Y_{n+1})}{f_0(Y_{n+1})} \,\middle|\, \mathcal{F}_n \right)
\end{aligned}$$

$$= X_n \mathsf{E}\left(\frac{f_1(Y_{n+1})}{f_0(Y_{n+1})}\right)$$

$$= X_n \int \frac{f_1(Y_{n+1})}{f_0(Y_{n+1})} \cdot f_0(Y_{n+1}) \partial Y_{n+1}$$

$$= X_n$$

Second and third equalities follow from Enhanced Scaling Rule and i.i.d. of Y_k's respectively.

(c)

$$\begin{aligned}
\mathsf{E}(X_{n+1} \mid \mathcal{F}_n) &= \frac{1}{n+3} \cdot \mathsf{E}\left[Y_{n+1} \;\middle|\; \frac{Y_0}{2}, \ldots, \frac{Y_n}{n+2}\right] \\
&= \frac{1}{n+3}\left[Y_n + \frac{Y_n}{n+2}\right] \\
&= \frac{Y_n}{n+2} = X_n
\end{aligned}$$

7

Lemma Time

$\star : S_k(t) \geq 0$

$\circ : S_k(t)$ is only nonzero for one value of k and zero everywhere else

$$\begin{aligned}
\sup_t |f_n(t) - f(t)| &= \sup_t \left| \sum_{k=n+1}^{\infty} a_k S_k(t) \right| \\
&\leq \sup_t \sum_{\ell=i}^{\infty} \sum_{k=2^\ell}^{2^{\ell+1}-1} |a_k| S_k(t) \quad \text{where } \ell < \log_2(n+1) \qquad (\text{by } \star) \\
&\leq \sup_t \sum_{\ell=i}^{\infty} \sum_{k=2^\ell}^{2^{\ell+1}-1} M(2^\ell)^\gamma S_k(t) \qquad \left(\text{since } k \text{ is at most } 2^\ell\right) \\
&= \sup_t \sum_{\ell=i}^{\infty} M(2^{\ell\gamma}) \sum_{k=2^\ell}^{2^{\ell+1}-1} S_k(t) \\
&\leq \sup_t \sum_{\ell=i}^{\infty} M(2^{\ell\gamma}) \left[2^{-\frac{1}{2}\cdot(\ell+1)}\right] \\
&\qquad \text{by } \circ \text{ and that the value is } \left[2^{-\frac{1}{2}\cdot(\ell+1)}\right]
\end{aligned}$$

$$= \sup_t \sum_{\ell=i}^{\infty} M' \cdot 2^{\ell[\gamma-1/2]}$$
$$\to 0 \quad \text{as } n \to \infty \text{ since } 0 \le \gamma < 1/2$$