36-703 Homework #3 Thursday 16 Feb 2006

1

LIF-ting

Fill out the calculations for the Lagrange Inversion Formula noted in the Notes on Generating Functions document.

2

Matrix Expansion

Following up on the pentagon problem, consider a finite set of nodes $S = \{1, \ldots, M\}$ and a process $X = (X_n)_{n \geq 0}$ that moves among them.

Assume that the distribution of X_0 is given by a row-vector $\boldsymbol{p}_0 = (p_{01}, \ldots, p_{0M})$, where $\mathsf{P}\{X_0 = i\} = p_{0i}$. (Of course, then $\boldsymbol{p}_0 \cdot \mathbf{1} = 1$.)

Assume that at any time $n \geq 0$, the conditional distribution of X_{n+1} given X_n is specified by

$$\mathsf{P}\{X_{n+1} = j \mid X_n = i\} = P_{ij},$$

where $i, j \in S$ and P is an $M \times M$ matrix.

Here, we will consider the row-vector-valued generating function

$$\boldsymbol{G}(z) = \sum_{n=0}^{\infty} \mathbf{p}_n z^n,$$

where $\boldsymbol{p}_n$ is the row-vector $(\mathsf{P}\{X_n = 1\}, \ldots, \mathsf{P}\{X_n = M\})$.

(a) Use the argument from the Pentagon example to show that

$$\mathbf{G}(z)(I - zP) = \boldsymbol{p}_0,$$

where I is the $M \times M$ identity

(b) Guess – and then make an argument to support – the form of the matrix valued generating function $F(z) = (I - zP)^{-1}$.

(c) Give an expression for the distribution of X_n in terms of P and $\boldsymbol{p}_0$.

3 Double Roots

In Example 1 on the class handout for 7 February, I developed a generating function strategy for a particular recurrence when the polynomial $az^2 + bz + c$ has two distinct, non-zero roots.

Comment in detail on whether this strategy can be adapted to the double-root case. If so, how? If not, is there another strategy that will work?

What does this imply about when one of two distinct roots is zero?

Is the double zero root case interesting? Why or why not?

4 Poisson, Naturally

(a) Let $N = (N_t)_{t \geq 0}$ be a counting process that satisfies:

1. $N_0 = 0$
2. N has stationary and independent increments
3. $\mathsf{P}\{N_h = 1\} = \lambda h + o(h)$, for some $\lambda > 0$.
4. $\mathsf{P}\{N_h > 1\} = o(h)$.

Show that N is a homogenous Poisson process with rate λ.

Note: Given sequences a_n and b_n, we say that $b_n = o(a_n)$ as $n \to \infty$ if $|b_n/a_n| \to 0$ as $n \to \infty$. Given functions $f(h)$ and $g(h)$, we say that $f(h) = o(g(h))$ as $h \to 0$ if $|f(h)/g(h)| \to 0$.

Hint: One approach is to divide up the interval t into many small sub-intervals and use the Poisson approximation to the Binomial.

(b) For a homogeneous Poisson process with rate λ, find the conditional distribution of S_1 (the time of first arrival) given $N_t = 1$.

(c) Following (b), find the joint distribution of S_1 and S_2 given $N_t = 2$.

(d) Following (c), find the joint distribution of $S_1, \ldots, S_n$ given $N_t = n$. How does this distribution relate to the distribution of a Uniform$\langle 0, t\rangle$ sample $U_1, \ldots, U_n$?

5

A Poisson Diet Plan

Suppose that the arrival of customers to a particular building conforms to a homogeneous Poisson process with rate $\lambda > 0$. Within the building are two restaurants – "Good Eats" and "Eat Good" – both of which serve the same kind of food. (Not a great business plan, I'll admit.) Thus, customers who enter the building choose at random between the two stores. Suppose that an entering customer eats at "Good Eats" with probability $0 < p < 1$ and at "Eat Good" with probability $q = 1 - p$.

Characterize the arrival processes of customers to each restaurant.

6

Martingales

In each of the following cases, show that the processes $X = (X_n)$ are martingales.

(a) Let Z be a real-valued random variable with $\mathsf{E}|Z| < \infty$. Let $Y_0, Y_, \ldots$ be a sequence of arbitrary random variables.

Define $X_n = \mathsf{E}(Z \mid Y_0, \ldots, Y_n)$.

(b) Let $Y_0, Y_1, \ldots$ be IID random variables. Let f_0 and f_1 be probability density functions that are positive on the common range of the Ys. Assume that f_0 is the true distribution of Y_1.

Define the likelihood ratio process

$$X_n = \frac{f_1(Y_0) f_1(Y_1) \cdots f_1(Y_n)}{f_0(Y_0) f_0(Y_0) \cdots f_0(Y_n)}.$$

(c) At time 0, a bucket contains one red and one blue ball. At each successive time, one ball is chosen at random (uniformly) from the bucket and that ball is returned along with one more of the same color. Let Y_n denote the number of blue balls at time n.

Define $X_n = \dfrac{Y_n}{n+2}$.

7

Lemma Time

Prove Lemma 8 from the 9 Feb class handout.

Optional extra: Prove Lemma 9.