36-703 **Homework #4** **Thursday 9 Mar 2006**

1 Preserving The Markov Property

G&S Section 6.2, exercises 7–9.

2 Card Trick

On the first day of class, I did a card trick. The participant picked a number k_0 from 1 to 9. The k_0th card in the well-shuffled deck becomes the participant's "key card". The value of that card 1 though 9 or 1 for a face card gives k_1. The k_1th subsequent card is the next key card and gives k_2. And so on. The participant is to remember the card (value and suit) of the most recent key card. When the deck runs out, the participant writes down the name (value and suit) of her most recent key card. Amazingly, I guess it – at least with high probability.

The key to this trick, excuse the pun, is that I use the first card as my initial key card and follow the same procedure as the participant.

Explain this in terms of Markov Chain theory. Use a computer or mathematical approximation, estimate the probability of this trick working. How can parameters of the trick (deck size, rules on face cards, etc.) might be tweaked to improve this probability?

3 Inventory Model

A newspaper uses one ton of newsprint every day, which it purchases from a local distributor. This distributor supplies the paper in one-ton rolls at the cheapest available price. Unfortunately, deliveries are erratic and on any particular day, the distribution of the number of rolls delivered is given by a pmf p. Assume the deliveries are IID across days. Assume also that the deliveries arrive in the evening and that one roll is used each morning.

If, come morning, the newspaper has no newsprint stored, it must purchase one roll at a high price from an emergency supplier. The regular supply still arrives in the evening.

Let X_n be the number of rolls that the newspaper has stored on the morning of day n, before the day's demand has been satisfied.

Show that $X = (X_n)_{n \geq 0}$ is a Markov chain and find its transition probabilities.

Let r be the price per roll from the distributor and c be the extra cost of an emergency roll. Describe in words how this model might be used by the newspaper to improve its situation. Feel free to introduce other parameters (e.g., storage costs, competitor distributions) as you see fit. In other words, tell a story that shows how calculations about a Markov chain can be used in practice. You don't need to actually do the calculations.

4

Periodic Limits

Suppose X is an irreducible, positive recurrent Markov chain on countable state-space $\mathcal{S}$ with period d.

Show that for any $s, s_0 \in \mathcal{S}$,

$$\lim_{n \to \infty} P^{nd}(s_0, s) = \frac{d}{M(s,s)}. \tag{1}$$

Explain why this should be so intuitively.

5

Queue Tip

Let $N = (N_n)_{n \geq 0}$ be the embedded Markov chain for a GI/M/1 queue that we discussed in class.

Let G be the arrival time distribution and let the service time distribution have an Exponential$\langle \lambda \rangle$ distribution.

Under what conditions does this have a stationary distribution? Find the stationary distribution under that assumption.

HINT: Guess at a specific form for π_k.

Hints as to an appropriate form are available on request, but try it first.

6

Finite Recurrence

Given an irreducible, aperiodic Markov chain on a finite state-space $\mathcal{S}$. Show that there exists an n such that all enteries in P^n are positive.

What can you say about the recurrence of such a chain? Does a stationary distribution always exist? Explain your answers.

7 — Markov Flu

Members of an indefinitely large population are either immune to the Markov Flu or are susceptible to it. Let S_n denote the number of susceptible members of the disease at time n and assume that $S_0 = 0$.

In each time period in which the disease is quiescent, assume that the number of susceptible individuals increases by 1 (i.e., $S_{n+1} = S_n + 1$). But at each time, there is a probability $0 < p < 1$ of a major outbreak in which case all susceptible individuals are stricken. The Markov Flu is nonlethal but is known to keep its victims up late at night with graphomania – scratching out obscure symbols on scraps of paper. After the bout, the Markov Flu confers immunity. Thus, at the time of an outbreak, S reaches 0.

Find the stationary distribution for (S_n).

8 — Murkiest Points

Describe the one point or idea in the material covered to date that you find the murkiest so far.

Then, grapple with that issue to try to understand it. Write a clear and coherent explanation of the idea that clears up the murk. Feel free to come see either me or Charles to help with your research. Feel free also to share your explanations with colleagues.