1

Lucky 13

Let $Y_0 = 0$ and Y_n for $n \geq 1$ be the sum of n rolls of a balanced, six-sided die.

Find

$$\lim_{n\to\infty} \mathsf{P}\{13 \backslash Y_n\},$$

where $j\backslash k$ ("j divides k") for integers k and $j > 0$ means that there is an integer m such that $k = jm$.

2

Circulant

Consider a Markov chain on $\mathcal{S} = \{0, \ldots, m\}$ with transition probabilities

$$P(i, j) = p_{j-i \bmod m+1},$$

where $p_0 > 0$ and $\sum_{i=0}^{m} p_i = 1$. (It might help to sketch out what this looks like in matrix form.)

Find the stationary distribution of such a chain.

3

Classes

Define a Markov chain on $\{1, \ldots, 6\}$ with transition probability matrix

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

Tell me everything you can about this chain.

4

Not our class's "class"

An assumption Sociologists might make is that a family's circumstances across successive generations satisfy the Markov property. Thus given a father's occupation, the son's would be independent of his grandfather's. Consider this simple (and cartoonish) Markov model for a family's social class on the state space $\mathcal{S} = \{\text{lower}, \text{middle}, \text{upper}\}$.

		Son's social class		
		lower	middle	upper
Father's	lower	0.40	0.50	0.10
social	middle	0.05	0.70	0.25
class	upper	0.05	0.50	0.45

What distribution of class – not our class's class but social class – does this model predict in the long-run?

5

Max Period

Let X be an irreducible Markov chain on a state space with M states. Show that the period of the chain must be no greater than M.

6

Second Order

Let X be an irreducible, aperiodic Markov chain on a state space $\mathcal{S}$. Define a new process Y on $\mathcal{S} \times \mathcal{S}$ by

$$Y_n = (X_n, X_{n+1})$$

for $n \geq 0$.

Is this new process a Markov chain? If so, is it irreducible? Is it aperiodic?

Explain your answers.

7

Null Limits

Let X be an irreducible, null recurrent Markov chain on countable state space $\mathcal{S}$.

Show that $\lim_{n\to\infty} P^n(s, s') = 0$ for any $s, s' \in \mathcal{S}$.

8

Transient Drift

Let Z be an irreducible, countable-state Markov chain on $\mathcal{S}$ with drift operator Δ. Then, Z is transient if and only if there exists an $s_0 \in \mathcal{S}$ and a bounded, nonconstant function V such that

$$\Delta V(s) = 0 \qquad \text{for } s \neq s_0.$$

Show this to be true.

Without loss of generality, you can take $\mathcal{S}$ to be the non-negative integers and $s_0 = 0$. Consider the hitting time of state 0. Remember what you need to establish transience for an irreducible chain.

9

Foster's: Australian for Drift

Give an intuitive explanation of Foster's drift criterion for recurrence.