36-703 Homework #6 Solutions Thursday 6 April 2006

Unless otherwise stated, for the remainder of the solutions, define

$$\mathcal{F}_m = \sigma(Y_0, \ldots, Y_m)$$

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Exercise 12.1.1

We will show $\mathsf{E}Y_m = \mathsf{E}Y_0$ using induction. $m = 0$ is obviously true. For base case $m = 1$: $\mathsf{E}Y_1 = \mathsf{E}[\mathsf{E}(Y_1 \mid Y_0)] = \mathsf{E}Y_0$. Now assume the property is true for $k < m$. Applying the Mighty Conditioning property and inductive hypothesis,

$$\mathsf{E}(Y_m) = \mathsf{E}\left[\mathsf{E}(Y_m \mid \mathcal{F}_{m-1})\right] = \mathsf{E}Y_{m-1} = \mathsf{E}Y_0$$

The second equality holds by the definition of martingale. Therefore, the property holds for all natural numbers. By definition of submartingale,

$$\mathsf{E}(Y_m) \leq \mathsf{E}\left[\mathsf{E}(Y_{m+1} \mid \mathcal{F}_m)\right] = \mathsf{E}Y_{m+1} \tag{1}$$

For the case of supermartingale, use its basic definition and flip the inequality in (1). Finally, use these inequalities in a similar inductive proof to show that $\mathsf{E}Y_m \geq \mathsf{E}Y_0$ and $\mathsf{E}Y_m \leq \mathsf{E}Y_0, \quad \forall m \in \mathbb{N}$.

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Exercise 12.1.2

By Mighty Conditioning Identity and definition of martingale

$$E(Y_{n+m} \mid \mathcal{F}_n) = \mathsf{E}\left[\mathsf{E}(Y_{n+m} \mid \mathcal{F}_{n+m-1}) \mid \mathcal{F}_n\right] = \mathsf{E}[Y_{n+m-1} \mid \mathcal{F}_n]$$

Now keep repeating this argument until

$$E(Y_{n+m} \mid \mathcal{F}_n) = E(Y_{n+1} \mid \mathcal{F}_n)$$

and the result follows by applying the definition of martingale.

3 Exercise 12.1.4

For a symmetric gambler's ruin, $S_{n+1} = S_n + X_{n+1}$ where

$$X_i \sim p(x) = \begin{cases} 1/2 & \text{if } x = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

So $E(X_i) = 0$ and $E(X_i^2) = 1 \ \ \forall i \geq 1$. Therefore,

$$\mathsf{E}[S_n \mid \sigma(S_0, \dots, S_{n-1})] = S_{n-1} + E[X_n \mid \sigma(S_0, \dots, S_{n-1})] = S_{n-1}$$

Now for $T_n = S_n^2 - n$ and $\mathcal{T} = \sigma(T_0, \dots, T_{n-1})$,

$$\begin{aligned} \mathsf{E}[T_n \mid \sigma(T_0, \dots, T_{n-1})] &= \mathsf{E}\left[(S_{n-1} + X_n)^2 - n \,\middle|\, \mathcal{T}\right] \\ &= \mathsf{E}\left[(S_{n-1}^2 - n + 1 - 1 + 2S_{n-1}X_n + X_n^2 \,\middle|\, \mathcal{T}\right] \\ &= \mathsf{E}\left[(S_{n-1}^2 - (n-1) - 1 + 2S_{n-1}X_n + X_n^2 \,\middle|\, \mathcal{T}\right] \\ &= \mathsf{E}\left[T_{n-1} + 2S_{n-1}X_n + X_n^2 - 1 \,\middle|\, \mathcal{T}\right] \\ &= T_{n-1} + 2S_{n-1} \cdot 0 + \mathsf{E}[X_n^2 \mid \mathcal{T}] - 1 = T_{n-1} \end{aligned}$$

Let p_k be the ruin probability given that we start from k and T be

$$T = \min\{n \geq 1 : S_n = 0 \text{ or } S_n = N\}$$

Assume $S_0 = k$. Optional stopping theorem says $\mathsf{E}(S_T) = \mathsf{E}(S_0) = k$. By this result,

$$\mathsf{E}(S_T) = N(1 - p_k) + 0 \cdot p_k = k \Rightarrow p_k = 1 - k/N$$

Similarly it is also true that $\mathsf{E}(S_T^2 - T) = k^2$. By this result,

$$\begin{aligned} k^2 = \mathsf{E}(S_T^2 - T) &= \mathsf{E}(S_T^2) - \mathsf{E}T \\ &= N^2(1 - p_k) + 0^2 \cdot p_k - \mathsf{E}T \\ &= N^2(1 - p_k) - \mathsf{E}T \end{aligned}$$

Next, do the Plug-N'-Chug with p_k (i.e. $1 - k/N$) and solve for $\mathsf{E}T$, whose solution turns out to be $k(N - k)$.

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Exercise 12.1.5

For $r \geq i, \mathsf{E}(Y_r \mid \mathcal{F}_i) = Y_i$ by Exercise 12.1.2. For $r \geq i$,

$$\Rightarrow \mathsf{E}(Y_r Y_i) = \mathsf{E}[\mathsf{E}(Y_r Y_i \mid \mathcal{F}_i)] \tag{2}$$

$$= \mathsf{E}[Y_i \mathsf{E}(Y_r \mid \mathcal{F}_i)] \quad \text{by Enhanced Scaling} \tag{3}$$

$$= \mathsf{E}(Y_i^2) \tag{4}$$

$\Rightarrow \mathsf{E}[(Y_k - Y_j)Y_i] = 0$ for $i \leq j \leq k$. For the next part, expand $(Y_k - Y_j)^2$ and notice that

$$\begin{aligned} \mathsf{E}(Y_k Y_j \mid \mathcal{F}_i) &= \mathsf{E}[\mathsf{E}(Y_k Y_j \mid \mathcal{F}_j) \mid \mathcal{F}_i] \\ &= \mathsf{E}[Y_j \mathsf{E}(Y_k \mid \mathcal{F}_j) \mid \mathcal{F}_i] \\ &= \mathsf{E}(Y_j^2 \mid \mathcal{F}_i) \end{aligned}$$

The three equalities follow from Mighty Conditioning Identity, Enhanced Scaling and Exercise 12.1.2. Expanding out $(Y_k - Y_j)^2$ as suggested:

$$\begin{aligned} \mathsf{E}\left[(Y_k - Y_j)^2 \,\middle|\, \mathcal{F}_i\right] &= \mathsf{E}\left(Y_k^2 \,\middle|\, \mathcal{F}_i\right) - 2\mathsf{E}(Y_k Y_j \mid \mathcal{F}_i) + E\left(Y_j^2 \,\middle|\, \mathcal{F}_i\right) \\ &= \mathsf{E}\left(Y_k^2 \,\middle|\, \mathcal{F}_i\right) - E\left(Y_j^2 \,\middle|\, \mathcal{F}_i\right) \end{aligned}$$

Take expectation on both sides of the previous claim. What we have is

$$0 \leq \mathsf{E}\left[(Y_k - Y_j)^2\right] = \mathsf{E}\left(Y_k^2\right) - E\left(Y_j^2\right)$$

We know the sequence $\{E\left(Y_n^2\right)\}$ is bounded by assumption and is non-decreasing. Therefore, it is a covergent sequence. Thus, as $k, j \to \infty$, $\mathsf{E}\left(Y_k^2\right) - E\left(Y_j^2\right) \to 0$ and so $\mathsf{E}\left[(Y_k - Y_j)^2\right] \to 0$. $\Rightarrow \{Y_n\}$ is Cauchy convergent in mean square. Using the result from Exercise 7.11.11, convergence in mean square follows directly.

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Exercise 12.1.6

Apply Jensen's inequality and use definition of martingale to establish that

$$\mu\{\mathsf{E}(Y_{n+1} \mid \mathcal{F}_n)\} = \mu\{Y_n\}$$

Finally, argue the three given functions are convex by observing its plot, second derivative (if it exists), or noting its epigraph, i.e. the set $\{(x, y) \mid y \geq f(x)\}$, is convex.

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Exercise 12.9.13

(a) This is a generalization of Problem 6 in Homework 3.

$$\mathsf{E}(Y_n \mid \mathcal{F}_{n-1}) = \frac{1}{n+r+b}\left([R_{n-1}+1]Y_{n-1} + R_{n-1}[1 - Y_{n-1}]\right)$$

$$= \frac{R_{n-1}}{n+r+b} + \frac{Y_{n-1}}{n+r+b} = Y_{n-1}$$

On the right hand side of second equality, plug in the value of Y_{n-1} in terms of R_{n-1}. The last equality results after a little algebra. Next, notice that $|Y_n| \leq 1$. This leads to that

$$|Y_n| I_{\{|Y_n| \geq a\}} \leq I_{\{1 \geq a\}}$$
$$\Rightarrow \sup_n \mathsf{E}\left(|Y_n| I_{\{|Y_n| \geq a\}}\right) \leq E[I_{\{1 \geq a\}}] \to 0$$

as $a \to \infty$. Therefore, Y_n is uniformly integrable. Then it follows that Y_n converges almost surely and in mean.

(b) Apply the optional stopping theorem. That is, $E(Y_T) = Y_0 = 1/2$. Then, observe $R_T = T$. Therefore,

$$1/2 = \mathsf{E}Y_T = \mathsf{E}\left(\frac{R_T}{T+2}\right)$$
$$= \mathsf{E}\left(\frac{T}{T+2}\right) = \mathsf{E}\left(\frac{T+2-2}{T+2}\right)$$
$$\Rightarrow \mathsf{E}\left(\frac{1}{T+2}\right) = \frac{1}{4}$$

(c) The maximal inequality gives that

$$\mathsf{P}\left(\max_{0 \leq i \leq n} Y_i \geq 3/4\right) \leq \frac{\mathsf{E}Y_n}{3/4} = \frac{\mathsf{E}Y_0}{3/4} = 2/3$$

The first equality came from Exercise 12.1.1.

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Exercise 12.9.18

Let $\mathcal{F} = \sigma(X_0, \ldots X_n)$.

$$\begin{aligned}
\mathsf{E}(X_{n+1} \mid \mathcal{F}) &= \mathsf{E}\left(\frac{R_{n+1}}{52-(n+1)} \,\Big|\, \mathcal{F}\right) \\
&= \frac{1}{52-(n+1)}\mathsf{E}\left(R_{n+1} \mid \mathcal{F}\right) \\
&= \frac{R_n}{52-(n+1)} \cdot (1 - X_n) + \frac{R_n - 1}{52-(n+1)} \cdot X_n = X_n
\end{aligned}$$

X_n is the proportion of red cards remaining in the deck of cards. Suppose the strategy is to call "Red Now" at some arbitrary time T. Apply the optional stopping theorem to show that $\mathsf{E}X_T = \mathsf{E}X_0 = 1/2$. Thus, for any $T \geq 0$, the probability of winning is 1/2.