1 -

Define $T_1 = \min\{n : Y_n \ge b\}$ and $T_2 = \min\{n > T_1 : Y_n \le a\}$ and Exercise 12.3.1 inductively that

$$T_{2k-1} = \min\{n > T_{2k-2} : Y_n \ge b\}$$
 and $T_{2k} = \min\{n > T_{2k-1} : Y_n \le a\}$

The interval $[T_{2k-1}, T_{2k}]$ is called a downcrossing of [a, b] and the number of downcrossing is the number of such intervals. The number of downcrossings by time n is $D_n(a, b; Y) = \max\{k : T_{2k} \leq n\}$, i.e. the number of downcrossings of [a, b] of the sequence Y_0, Y_1, \ldots, Y_n .

(a) Between each pair of downcrossings of [a, b], there must be an upcrossing and vice versa. Therefore

$$|D_n(a,b;Y) - U_n(a,b;Y)| \le 1 \quad \forall n \in \mathbb{N}$$

(b) First, we need to show that T_k are indeed stopping times. T_1 is a stopping time because $\{T_1 \leq n\} = \bigcup_{j=0}^n \{Y_j \geq b\} \in \mathcal{F}_n$. Assume T_{2k-1} is a stopping time (inductive hypothesis) and we arrive at

$$\{T_{2k} \le n\} = \bigcup_{j=0}^{n} \left(\{T_{2k-1} \le j-1\} \cap \{Y_j \le a\}\right) \in \mathcal{F}_n$$

So by induction on k, it follows that T_k is a stopping time. Define for $j \in \mathbb{N}, A_j = \{\omega : T_{2k-1}(\omega) < j \leq T_{2k}(\omega) \text{ for some k}\}$ and

$$Z_n = \sum_{j=1}^n I_{A_j} (Y_j - Y_{j-1}), \quad n \in \mathbb{N}$$

 I_{A_j} is measurable with respect to \mathcal{F}_{j-1} because

$$I_{A_j}^{-1}(\{1\}) = \bigcup_k \left(\{T_{2k-1} \le j-1\} \setminus \{T_{2k} \le j-1\} \right) \in \mathcal{F}_{j-1}$$
(1)

The \in relation follows from our proof that T_k that we defined are stopping times. Using (1), we have that

$$\mathsf{E}(Z_n - Z_{n-1}) = \mathsf{E}\left[\mathsf{E}\left(I_{A_n}(Y_n - Y_{n-1}) \mid \mathcal{F}_{n-1}\right)\right]$$
$$= \mathsf{E}\left\{I_{A_n}\left[\mathsf{E}(Y_n \mid \mathcal{F}_{n-1}) - Y_{n-1}\right]\right\} \ge 0$$

Note the inequality follows from the definition of submartingale. Then, we have that

$$\mathsf{E}(Z_n) \ge E(Z_{n-1}) \ge \ldots \ge E(Z_0) = 0$$

Observe that $Z_n \leq -(b-a)D_n(a,b;Y) + (Y_n-b)^+$. Taking expectation on both sides,

$$(b-a)\mathsf{E}[D_n(a,b;Y)] \le \mathsf{E}\left[(Y_n-b)^+\right] - \mathsf{E}Z_n$$

and the result follows from our proof of that $\mathsf{E}Z_n \ge 0$.

2 -

If Y is a supermartingale, then -Y is a submartingale. Upcrossings Exercise 12.3.2 of [a, b] by Y correspond to downcrossings of [-b, -a] of -Y.

By Exercise 12.3.1,

$$\mathsf{E}[U_n(a,b;Y)] = \mathsf{E}[D_n(-b,-a;-Y)]$$
$$\leq \frac{\mathsf{E}[(a-Y_n)^+]}{b-a}$$
$$= \frac{\mathsf{E}[(Y_n-a)^-]}{b-a}$$

Note that if $a, Y_n \ge 0$, then $(Y_n - a)^- \le a$

3 -

The condition ψ has to satisfy means that the sequence $\{\psi(X_n)\}$ is a bounded supermartingale. By the supermartingale convergence theorem, it converges almost surely to some limit. The chain X_n defined on state space \mathcal{S} is irreducible and recurrent. Therefore, each state is visited infinitely often a.s. But the codomain of ψ is this same state space. Thus, if ψ is defined so that it takes on more than 1 value in the codomain, these values will be visited infinitely often. This contradicts the a.s. convergence, unless ψ returns only one value, in which case ψ is a constant function. Exercise 12.3.3

 Y_n is the sum of independent variables with zero means $\Rightarrow Y$ is a Exercise 12.3.4 martingale. Also note that

$$\sum_{n=1}^{\infty} \mathsf{P}(Z_n \neq 0) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

By the Borel-Cantelli Lemma

$$\mathsf{P}(Z_n \neq 0 \text{ infinitely often}) = 0$$

in other words, $Z_n = 0$ almost surely. Therefore, the partial sum of Y_n converges a.s. to some finite limit as n approaches infinity.

By induction, one can show that $a_n = 5a_{n-1}$ for $n \ge 3$. Therefore, $a_n = 8 \cdot 5^{n-2}$ for $n \ge 3$. However note that

$$|Y_n| \ge \frac{1}{2}a_n$$
 if and only if $|Z_n| = a_n$

Because of Markov's Inequality,

$$E(|Y_n|) \ge \frac{1}{2} a_n \mathsf{P}\left(|Y_n| \ge \frac{1}{2} a_n\right)$$

= $\frac{1}{2} a_n \mathsf{P}(|Z_n| = a_n) = \frac{a_n}{2n^2} = \frac{8 \cdot 5^{n-2}}{2n^2} \to \infty$

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Generally for questions like these, it helps to re-express the events using complements, unions and intersections. Then, apply the properties of σ -fields and filtration to reach the result. Let \mathcal{F} be the filtration and $\mathcal{F}_n \in \mathcal{F}$. In this problem, Exercise 12.4.1

$$\{T_1 + T_2 = n\} = \bigcup_{k=0}^{n} (\{T_1 = k\} \cap \{T_2 = n - k\})$$
(2)

$$\{\max\{T_1, T_2\} \le n\} = \{T_1 \le n\} \cap \{T_2 = n\}$$
(3)

$$\{\min\{T_1, T_2\} \le n\} = \{T_1 \le n\} \cup \{T_2 = n\}$$
(4)

Now using properties of filtration and σ -fields, the events on the RHS of the equality in (2), (3) and (4) are also in \mathcal{F}_n .

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and $S_n = \sum_{k=1}^n X_k$ be given. We can re-express

Exercise 12.4.2

the event as

$$\{N(t) + 1 = n\} = \{S_{n-1} \le t\} \cap \{S_n > t\}$$

Use the same argument made in 12.4.1 and we have that

$$\{N(t)+1=n\}\in\mathcal{F}_n$$

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Similar to Exercise 12.3.1, we define a new random variable Z_n

Exercise 12.4.5

$$Z_n = \begin{cases} \sum_{k=1}^n Y_k - Y_{k-1} & \text{if for each } k, S < k \le T \\ 0 & \text{otherwise} \end{cases}$$

Similar to (1), one can prove that $I_{\{S < n \leq T\}}$ is measurable with respect to \mathcal{F}_{n-1} . By the definition of Z_n , $Z_N = Y_T - Y_S$. What remains to be shown now is that $\mathsf{E}Z_n \ge 0$. If we have that, then $\mathsf{E}Y_T \ge \mathsf{E}Y_S$. To show that, note

$$\mathsf{E}(Z_n - Z_{n-1}) = \mathsf{E}\left\{I_{\{S < n \le T\}}(Y_n - Y_{n-1})\right\}$$
$$= \mathsf{E}\left\{I_{\{S < n \le T\}}\mathsf{E}(Y_n - Y_{n-1} \mid \mathcal{F}_{n-1})\right\}$$
$$\ge 0 \quad \text{by definition of submartingale}$$

Second equality comes from applying Mighty Conditioning Identity, measurability of the indicator function with respect to \mathcal{F}_{n-1} , and Enhanced Scaling. Therefore,

$$\mathsf{E}Z_N \ge \mathsf{E}Z_{N-1} \ge \dots \ge \mathsf{E}Z_0 = 0$$

Applying definition of De Moivre's Martingale and maximal inequal- Exercise 12.4.6 ity,

$$\mathsf{P}\left(\max_{0\leq m\leq n} S_m \geq x\right) = \mathsf{P}\left(\max_{0\leq m\leq n} Y_m \geq (q/p)^x\right) \leq (p/q)^x$$

Let n go to infinity, the limit becomes $\sup_m S_m$. Now we can use probability rule for discrete nonnegative random variables: Suppose N is non-negative and discrete random variable. Then

$$\mathsf{E}N = \sum_{i=0}^{\infty} \mathsf{P}(N > i)$$

You've used this fact in Homework 1. To see why this is true, refer to Miscellaneous Results section for its proof. Therefore, it follows from the rule

$$\mathsf{E}\left(\sup_{m} S_{m}\right) = \sum_{x=0}^{\infty} \mathsf{P}\left(\sup_{m} S_{m} > x\right) = \sum_{x=1}^{\infty} \mathsf{P}\left(\sup_{m} S_{m} \ge x\right)$$
$$\leq \frac{p}{q} \sum_{x=1}^{\infty} \left(\frac{p}{q}\right)^{x-1}$$
$$= \frac{p}{q-p}$$

The equality follows from the result in Exercise 5.3.1. See Miscellaneous Results section for proof of the result.

(a)

Exercise 12.4.7

(ii) Assume $A \in \mathcal{F}_T$. Note that

(i) $\Omega \cap \{T \leq n\} = \{T \leq n\} \in \mathcal{F}_n \Rightarrow \Omega \in \mathcal{F}_T$

$$A^C \cap \{T \le n\} = \{T \le n\} \cap (A \cap \{T \le n\})^C \in \mathcal{F}_n$$

since members of a σ -field are closed under complement and intersection. It follows that $A \in \mathcal{F}_T$.

(iii) Now assume that $A_1, A_2, \ldots \in \mathcal{F}_T$. Then,

$$\left(\bigcup_{i=1}^{\infty} A_i\right) \bigcap \{T \le n\} = \bigcup_{i=1}^{\infty} (A_i \cap \{T \le n\}) \in \mathcal{F}_n$$

$$\Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right) \in \mathcal{F}_T \quad \text{using the same argument in (ii)}$$

Since (i), (ii), and (iii) are satisfied, \mathcal{F}_T is a σ -field. Now to show that T is measurable with respect to \mathcal{F}_T , we need to show that for each integer m, $\{T \leq m\} \in \mathcal{F}_T$. For any integer m,

$$\{T \le m\} \cap \{T \le n\} = \begin{cases} \{T \le n\} & \text{if } m > n\\ \\ \{T \le m\} & \text{if } m \le n \end{cases}$$

In both cases, they are in \mathcal{F}_n . The second case is true by definition of filtration. So T is measurable with respect to \mathcal{F}_T .

(b) Again, it helps to write

$$(A \cap \{S \le T\}) \cap \{T \le n\}$$

as union and intersection of events in \mathcal{F}_n . It turns out that

$$(A \cap \{S \le T\}) \cap \{T \le n\} = \bigcup_{m=0}^{n} (A \cap \{S \le m\}) \cap \{T \le m\}$$

Since for each m in the union, $m \leq n$, it follows from the definition of filtration that for each m

$$\{T \le m\} \in \mathcal{F}_n \quad \text{and} \quad A \cap \{S \le m\} \in \mathcal{F}_n$$

By the properties of σ -field, $A \cap \{S \leq T\} \in \mathcal{F}_T$.

(c) We know $\{S \leq T\} = \Omega$. Now use your best friend: Mr. Plug-N'-Chug to plug and chug in the value $\{S \leq T\} = \Omega$ into the result in (b) and the claim is proven.

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10 -

We have seen that $Y_n = \mathsf{E}(Y_\infty \mid \mathcal{F}_n)$. In terms of definition of conditional expectation, this means that

$$\mathsf{E}(Y_n I_C) = \mathsf{E}(Y_\infty I_C) \quad \forall \ C \in \mathcal{F}_n \tag{5}$$

Recall the formal definition of conditional expectation: Given a function h and that it is \mathcal{F} -measurable, $h = \mathsf{E}(X \mid \mathcal{F})$ almost surely if:

$$\mathsf{E}(hI_C) = \mathsf{E}(XI_C) \ \forall \ C \in \mathcal{F}$$

In the context of this problem, we need to prove

$$\mathsf{E}(Y_T I_C) = \mathsf{E}(Y_\infty I_C) \quad \forall \ C \in \mathcal{F}_T$$

Let $C \in \mathcal{F}_T \Rightarrow C \cap \{T = n\} \in \mathcal{F}_n$

$$Y_T I_C = Y_T I_C \sum_n I_{\{T=n\}}$$
$$= \sum_n Y_T I_{C \cap \{T=n\}}$$
$$= \sum_n Y_n I_{C \cap \{T=n\}}$$

$$\Rightarrow \mathsf{E}(Y_T I_C) = \sum_n \mathsf{E}\left(Y_n I_C \cap_{\{T=n\}}\right)$$
$$= \sum_n \mathsf{E}\left(Y_\infty I_C \cap_{\{T=n\}}\right) \quad \text{by (5)}$$
$$= \mathsf{E}\left(Y_\infty I_C \sum_n I_{\{T=n\}}\right) = \mathsf{E}(Y_\infty I_C)$$

For the next claim, We need to use the fact that $\mathcal{F}_S \subseteq \mathcal{F}_T$ proven in Exercise 12.4.7. Then using the Mighty Conditioning Identity,

$$Y_S = \mathsf{E}(Y_{\infty} \mid \mathcal{F}_T)$$
$$= \mathsf{E} \{ E(Y_{\infty} \mid \mathcal{F}_T) \mid \mathcal{F}_S \}$$
$$= \mathsf{E}(Y_T \mid \mathcal{F}_S) \quad \text{by Exercise 12.1.2}$$

Similar to Exercise 12.1.4 in the last homework. The only difference is Exercise 12.5.4 that you need to check the conditions of the optional stopping theorem before applying it. In the previous homework, many people either did not check the conditions or they simply said $EX_T = EX_0$ is a direct result of a martingale, which is not true when T is random. To see the difference, compare Exercise 12.1.1 with the Optional Stopping Theorem.

1. Proof of: Suppose N is non-negative and discrete random variable. Then

$$\mathsf{E}N = \sum_{k=0}^{\infty} \mathsf{P}(N > k)$$

$$\mathsf{E}N = \sum_{n=0}^{\infty} n \mathsf{P}(N=n) = \sum_{n=1}^{\infty} n \mathsf{P}(N=n)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathsf{P}(N=n)$$

$$= \sum_{k=1}^{\infty} \mathsf{P}(N \ge k)$$

$$= \sum_{k=0}^{\infty} \mathsf{P}(N > k)$$

2. For the continuous case,

$$\begin{split} \int_0^\infty (1 - F_N(n)) dn &= \int_0^\infty \mathsf{P}(N > n) dn \\ &= \int_0^\infty \int_n^\infty f_N(y) dy dn \\ &= \int_0^\infty \int_0^y dn f_N(y) dy \\ &= \int_0^\infty y f_N(y) dy = \mathsf{E}N \quad \dagger \end{split}$$

3. Proof of the claim posed in Exercise 5.3.1.

For a simple random walk S_n with $S_0 = 0$ and p = 1 - q < 1/2, $M = \max\{S_n : n \ge 0\}$ satisfies

$$\mathsf{P}(M \ge m) = (p/q)^m \text{ for } m \ge 0$$

Let A_k be the event that the random walk ever reaches k. So, $A_k \supseteq A_{k+1}$ if $k \ge 0$. Therefore,

$$\mathsf{P}(M \ge m) = \mathsf{P}(A_m) = \mathsf{P}(A_0) \prod_{k=0}^{m-1} \mathsf{P}(A_{k+1} \mid A_k)$$
$$= (p/q)^m$$

since $\mathsf{P}(A_{k+1} \mid A_k) = \mathsf{P}(A_1 \mid A_0) = p/q$ (See Corollary 5.3.6). \dagger

Miscellaneous Results