36-703 Homework #8 Solutions Friday 28 April 2006

1

Haar-d Made Easy

(a) This is a direct result of Cauchy-Schwarz inequality.

(b) The basic idea is to see that M_J is a martingale. Then apply the martingale convergence theorem. So, M_J converges to some limit, M_∞. M_J is a martingale since the definition of the Haar functions implies that

$$\mathsf{E}(H_{jk}(U) \mid \mathcal{F}_{j-1}) = 0$$

Now we proceed to show $M_\infty = f(U)$. Observe that

$$\begin{aligned}\mathsf{E}(f(U) \mid \mathcal{F}_J) &= \mathsf{E}(f(U) - M_J + M_J \mid \mathcal{F}_J)\\ &= \mathsf{E}(f(U) - M_J \mid \mathcal{F}_J) + M_J\end{aligned}$$

But $\mathsf{E}(f(U) - M_J \mid \mathcal{F}_J) = 0$ since $f(U) - M_J$ is orthonormal to M_J. Therefore, we have that

$$M_J = \mathsf{E}(f(U) \mid \mathcal{F}_J) \tag{1}$$

Finally, apply Lemma 12.3.11 to get that $M_J \to f(U)$ as $J \to \infty$.

(c) We will now show M_J is uniformly integrable and L_1 convergence follows immediately. To prove that M_J is uniformly integrable, combine the results from part (a) and (1) to use the result from Example 7.10.13. Then, conclude that M_J is uniformly integrable.

One can also try to show that $\mathsf{E}|M^2|$ is finite.

(d) Square out the integrand. Then using basic calculus and facts about orthonormality, you would find that the quantity on LHS is nonincreasing and is bounded below by 0. Thus, the integral converges to 0.

2

Poisson Martingale

First, $\mathsf{E}|M_t| \le \mathsf{E}|N_t| + |\lambda t| = \mathsf{E}N_t + \lambda t = 2\lambda t < \infty$

Next, using basic properties of conditional expectation,

$$\begin{aligned}\mathsf{E}(M_t \mid \mathcal{F}_s) &= \mathsf{E}(N_s + N_t - N_s - \lambda t \mid \mathcal{F}_s)\\ &= N_s + \mathsf{E}(N_t - N_s \mid \mathcal{F}_s) - \lambda t\\ &= N_s + \lambda t - \lambda s - \lambda t = N_s - \lambda s = M_s\end{aligned}$$

3

Thinned Renewal Process

Let $T_0 = 0$ and T_k for $k \geq 1$ be the interarrival time of the k^{th} acceptance of an arrival from N. Then, $T_k \sim \text{Geometric}(p)$. We can write for $k \geq 1$

$$W_k = \sum_{i=s_{k-1}+1}^{s_k} Y_i \quad \text{where } s_k = \sum_{j=0}^{k} T_j \quad \text{and}$$

$$\widetilde{S}_n = S_0 + \sum_{k=1}^{n} W_k$$

Finally, $\widetilde{N}_t = \sup\{n : \widetilde{S}_n \leq t\}$ defines the thinned renewal process based on the formal definition of a renewal process.

We can rewrite the thinned process as

$$\widetilde{N}_t = \sum_{k=1}^{\infty} 1\{S_k \leq t\} 1\{U_k \leq p\}$$

Then,

$$\widetilde{m}(t) = \mathsf{E}\left(\sum_{k=1}^{\infty} 1\{S_k \leq t\} 1\{U_k \leq p\}\right) = p \cdot \mathsf{E}\left(\sum_{k=1}^{\infty} 1\{S_k \leq t\}\right) = p \cdot m(t)$$

4

Inhomogeneous Construction

(a) For $s_1 < t_1 < T, N_t - N_s$ depends on $\widetilde{N}_{t_1} - \widetilde{N}_{s_1}$ and $\lambda(r)$ where $s_1 < r < t_1$. Similarly, the same applies for $s_2 < t_2 < T$ where $s_1 < t_1 < s_2 < t_2$. But by definition, $\widetilde{N}_{t_1} - \widetilde{N}_{s_1}$ and $\widetilde{N}_{t_2} - \widetilde{N}_{s_2}$ are independent. Now think of keeping or discarding the arrivals as generating i.i.d. uniform random variables on $[0, 1]$, U_k and comparing that to $\lambda(T_k)/\lambda_{\max}$. Then, by independence, $N_{t_1} - N_{s_1}$ and $N_{t_2} - N_{s_2}$ are independent.

(b) For $t < T$, N_t depends on $\widetilde{N}_t$. To find $\mathsf{E}N_t$, it helps to condition on $\widetilde{N}_t$.

$$\mathsf{E}N_t = \sum_{k=1}^{\infty} \mathsf{E}\left(N_t \,\middle|\, \widetilde{N}_t = k\right) \mathsf{P}\left(\widetilde{N}_t = k\right)$$

Replace N_t with $\sum_{i=1}^{\infty} 1\{T_i \leq t\} 1\{U_i = 1\}$ for $U_i \sim \text{Uniform}(0, 1)$. Then,

$$\mathsf{E}N_t = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \int_0^t \frac{\lambda(r)}{\lambda_{\max}} dr \cdot \mathsf{P}\left(\widetilde{N}_t = k\right)$$

$$= \sum_{i=1}^{\infty} \mathsf{P}\left(\widetilde{N}_t \geq i\right) \int_0^t \frac{\lambda(r)}{\lambda_{\max}} dr = \frac{\mathsf{E}\widetilde{N}_t}{\lambda_{\max}} \int_0^t \lambda(r) dr$$
$$= \int_0^t \lambda(r) dr$$

Use the same argument for $\mathsf{E}N_s$ and combine the two expected values.

(c) It's clear $N_0 \leq \widetilde{N}_0 = 0$. So, $N_0 = 0$. Independent increments follow from (a). Also,

$$\mathsf{P}(N_{t+h} - N_t \geq 2) \leq \mathsf{P}\left(\widetilde{N}_{t+h} - \widetilde{N}_t \geq 2\right) = o(h).$$

Finally,

$$\mathsf{P}(N_{t+h} - N_t = 1) = \sum_{k=1}^{\infty} \mathsf{P}\left(N_{t+h} - N_t = 1 \,\middle|\, \widetilde{N}_{t+h} - \widetilde{N}_t = k\right) \mathsf{P}\left(\widetilde{N}_{t+h} - \widetilde{N}_t = k\right)$$
$$= \sum_{k=1}^{\infty} \binom{k}{1} \left(\frac{\lambda(t)}{\lambda_{\max}}\right) \left(1 - \frac{\lambda(t)}{\lambda_{\max}}\right)^{n-1} \frac{\exp\{-\lambda_{\max} \cdot h\}(\lambda_{\max} \cdot h)^k}{k!}$$
$$= \exp\{-\lambda_{\max} \cdot h\} \lambda(t) h$$
$$= \lambda(t) h + o(h)$$

The last equality comes from using the approximation for small a,

$$e^{-a} \approx 1 - a$$

5

Compound Poisson Process

This is a specific type of the random sum problem. The random sum problem states that for random N defined on set of positive integers and X_i from a random sample,

$$\mathsf{E}\left[\sum_{i=1}^{N} X_i\right] = \mathsf{E}(N)\mathsf{E}(X)$$

and that

$$\mathsf{Var}\left[\sum_{i=1}^{N} X_i\right] = \mathsf{E}(N)\,\mathsf{Var}(X) + (\mathsf{E}X)^2\,\mathsf{Var}(N)$$

Proof of this involves rule of iterated expectation and will be left as an exercise for you. Using these results, we have

$$\mathsf{E}N_t = \lambda t \mathsf{E}Y \quad \text{and} \quad \mathsf{Var}\, N_t = \lambda t \mathsf{E}Y^2$$

One can also find the first and second moments using generating functions. Using Theorem 25 in Chapter 5.1, we have $\mathsf{G}_{N_t}(s) = \mathsf{G}_{N_t^\circ}(\mathsf{G}_Y(s))$. Finally use the fact that

$$\mathsf{E}[N_t(N_t - 1)\cdots(N_t - k + 1)] = \mathsf{G}_{N_t}^{(k)}(1)$$

to derive the first and second moments.

6 — Time Change

Similar to the Inhomogeneous Construction problem, this is another way to construct an inhomogeneous Poisson Process. This is only possible if the described $m(\cdot)$ is not too difficult to compute.
Now notice $M_0 = N_{m^{-1}(0)} = N_0 = 0$. Independent increments follow from the fact that $m(\cdot)$ is monotone. Finally notice that

$$\begin{aligned}
\mathsf{P}(M_t - M_s = k) &= \mathsf{P}(N_{m^{-1}(t)} - N_{m^{-1}(s)} = k) \\
&= \frac{1}{k!} \exp\left\{m(m^{-1}(t)) - m(m^{-1}(s))\right\} \left[m(m^{-1}(t)) - m(m^{-1}(s))\right]^k \\
&= \frac{e^{-(t-s)}(t-s)^k}{k!}
\end{aligned}$$

Thus, $M_t - M_s \sim \text{Poisson}(t - s)$ and has unit rate.