1

Haar-d Made Easy

Define $H = 1_{(0,1/2]} - 1_{(1/2,1]}$. Then let

$$H_{jk}(t) = 2^{j/2} H(2^j t - k).$$

Then, $H_0 = 1_{(0,1]}$ and H_{jk} for $j \geq 0$, $k = 0, \ldots, 2^j - 1$ is called the *Haar basis*.

These functions form a complete orthonormal basis for $L^2(0,1)$. For orthonormality, note that $\int H_0^2 = 1$, $\int H_{jk} H_{j'k'} = \delta_{jj'} \delta_{kk'}$, and $\int H_{jk} = 0$. We also have that $\alpha H_0 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \beta_{jk} H_{jk}$ gives a representation for all piecewise constant functions on dyadic intervals of length 2^{-J}.

Now suppose that f is a function on $[0,1]$ with $\int_0^1 f^2 < \infty$.

(a) Show that $\int_0^1 |f| < \infty$.

Define $a = \int_0^1 f H_0 = \int_0^1 f$ and define $b_{jk} = \int_0^1 f H_{jk}$. Let U be a Uniform$\langle 0, 1 \rangle$ random variable.

Define a stochastic process $M = (M_J)_{J \geq 0}$ by $M_0 = a H_0(U)$ and for $n > 0$

$$M_J = a H_0(U) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} b_{jk} H_{jk}(U),$$

Notice that the number of terms increases with each J.

(b) M_n converges to $f(U)$ almost surely.

(c) Show also that

$$\int_0^1 \left| f - a H_0 - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} b_{jk} H_{jk} \right| \to 0$$

(d) Show that the Haar basis is complete for $L^2(0,1)$. That is, for any $f \in L^2(0,1)$ with a and b_{jk} defined as above

$$\int_0^1 \left| f - a H_0 - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} b_{jk} H_{jk} \right|^2 \to 0$$

as $J \to \infty$.

2

Poisson Martingale

Let N_t be a homogeneous Poisson process with rate $\lambda > 0$. Define

$$M_t = N_t - \lambda t, \quad t \geq 0,$$

and let $\mathcal{F}_t$ be the history of N up to time t for each $t \geq 0$.

Show that $M = (M_t)_{t \geq 0}$ is a (continuous-time) Martingale.

3

Thinned Renewal Process

Let N denote a renewal process with inter-renewal time distribution F. Let $(S_n)_{n \geq 1}$ denote the renewal times and let m denote the renewal function.

Suppose we "thin" this process as follows. For each S_n, we independently generate a Bernoulli$\langle p \rangle$ random variable U_n and keep S_n if $U_n = 1$. Otherwise, we delete S_n. Assume that the $(U_n)_{n \geq 1}$ sequence is independent of the renewal process (i.e., the S_ns).

Let $\widetilde{N}$ denote the counting process generated by the retained S_ns. Show that this is still a renewal process and find its renewal function in terms of m and p.

4

Inhomogeneous Construction

Suppose $0 \leq \lambda(u) \leq \lambda_{\max} < \infty$ for $u \in [0, T]$. Consider the following simulation method to generate an inhomogeneous Poisson process with intensity function λ over $[0, T]$.

Let $\widetilde{N}$ denote a homogeneous Poisson process with rate $\lambda_{\max}$. Let $0 < T_1 < T_2 < \cdots \leq T$ denote the points yielded by that process. Then independently for each $k = 1, \ldots, T_{N_T}$, retain point T_k with probability $\lambda(T_k)/\lambda_{\max}$. Otherwise, delete it.

Let N (at least for $0 \leq t \leq T$) denote the corresponding counting process (or equivalently random measure).

(a) Show that the resulting process (on $[0, T]$) has independent increments.

(b) Show that $\mathsf{E}(N_t - N_s) = \int_s^t \lambda(u)\, du$ for $0 \leq s < t \leq T$.

(c) Show that N_t is an inhomogeneous Poisson process with intensity function λ on $[0, T]$.

5 — Compound Poisson Process

Let $Y = (Y_n)_{n\geq 1}$ denote an IID sequence of $\mathbb{Z}_\oplus$-valued random variables with PGF G_Y.

Let N° denote a homogeneous Poisson process with rate $\lambda \geq 0$, independent of the Ys.

Define

$$N_t = \sum_{n=1}^{N_t^\circ} Y_n.$$

N is called a *compound Poisson process.*

Find the PGF, expected value, and variance of N_t.

6 — Time Change

Let N be an inhomogeneous Poisson process with intensity function $\lambda > 0$. Let

$$m(t) = \int_0^t \lambda(s)\, ds.$$

Because $\lambda(u) > 0$ for all $u \geq 0$, m is a monotone, and thus invertible function. Let m^{-1} denote the inverse.

Define $M_t = N_{m^{-1}(t)}$. That is, we've used m^{-1} to change time.

Show that M is a homogeneous Poisson process with rate 1.