36-703 Solution to final homework!! Friday 5 May 2006

Please report any typos immediately for a timely response.

1

Question 1

Call $\mathsf{E}(W_{t_n} \mid W_{t_{n-1}} \text{ near } u_{n-1}, \ldots, \ldots, W_{t_1} \text{ near } u_1)$ "LHS" for short. Add the quantity $W_{t_{n-1}} - W_{t_{n-1}}$ into the expected value of the LHS. Apply independent increments property of Weiner process and we get

$$\text{LHS} = \mathsf{E}(W_{t_n} - W_{t_{n-1}} \mid W_{t_{n-1}} \text{ near } u_{n-1}) + W_{t_{n-1}}$$

Finally apply linearity of expectation and the result follows.

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Question 2

Recall that for a Weiner process W_t

$$W_t - W_s \sim \text{ Gaussian}(0, t-s) \tag{1}$$

To prove the finite-ness of $\mathsf{E}|W_t|$, it's easy to derive using integration that

$$\mathsf{E}|W_t| = \sqrt{\frac{2t}{\pi}} < \infty$$

Next, by independent increments and the fact that $W_t - W_s$ is gaussian with mean 0, $\mathsf{E}(W_t - W_s \mid \mathcal{F}_s) = 0$. Therefore,

$$\mathsf{E}(W_t \mid \mathcal{F}_s) = \mathsf{E}(W_t - W_s \mid \mathcal{F}_s) + \mathsf{E}(W_s \mid \mathcal{F}_s) = 0 + W_s = W_s$$

and so W_t is a martingale.
Now look at U_t:

$$\mathsf{E}|U_t| \leq \mathsf{E}|W_t|^2 + t = \mathsf{Var}(W_t) + t = 2t < \infty$$

Also, for $s < t$

$$\begin{aligned}
\mathsf{E}(U_t \mid \mathcal{F}_s) &= \mathsf{E}(W_t^2 - t \mid \mathcal{F}_s) \\
&= \mathsf{E}[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 \mid \mathcal{F}_s) - t \\
&= \mathsf{E}[(W_t - W_s)^2] + 2W_s\mathsf{E}(W_t - W_s \mid \mathcal{F}_s) + W_s^2 - t \\
&= \mathsf{Var}(W_t - W_s) + 0 + W_s^2 - t \\
&= (t-s) + W_s^2 - t = U_s
\end{aligned}$$

The last two equalities follow from (1). Finally for V_t, notice that

$$\mathsf{E}|V_t| = \exp\{2^{-1}\lambda^2 t\}\mathsf{E}(\exp\{\lambda W_t\}) = 1 < \infty$$

The last equality comes from the fact that the expected value is the MGF of Gaussian$(0, t)$. Plug-N'-Chug in the exact value of the MGF and they evalute to 1. Finally, for $s < t$,

$$\begin{aligned}
\mathsf{E}(V_t \mid \mathcal{F}_s) &= e^{2^{-1}\lambda^2 t}\mathsf{E}(\exp\{\lambda(W_t - W_s + W_s)\} \mid \mathcal{F}_s) \\
&= e^{2^{-1}\lambda^2 t}\mathsf{E}(\exp\{\lambda(W_t - W_s)\} \mid \mathcal{F}_S)\mathsf{E}(\exp\{\lambda W_s\} \mid \mathcal{F}_s) \\
&= e^{2^{-1}\lambda^2 t}e^{2^{-1}\lambda^2(t-s)}\mathsf{E}(\exp\{\lambda W_s\} \mid \mathcal{F}_s) \\
&= e^{W_s - \frac{1}{2}\lambda^2 s} = V_s
\end{aligned}$$

3

Question 3

Let $T = \min\{t \geq 0 : W_t \in [a, b]\}$, p_b be the probability of hitting b before a, and similarly for p_a. Verify yourself the optional stopping theorem can be used (most importantly the uniformly integrability property). Thus,

$$\mathsf{E}(W_T) = bp_b + a(1 - p_b) = \mathsf{E}(W_0) = 0 \Rightarrow p_b = \frac{-a}{b - a}$$

Similarly, $p_a = \dfrac{b}{b - a}$

Next, apply optional stopping theorem one more time to see that

$$\mathsf{E}U_T = \mathsf{E}U_0 = 0 \Rightarrow \mathsf{E}(W_T^2) = \mathsf{E}T$$

Note that $\mathsf{E}T = \mathsf{E}(W_T^2) \leq a^2 + b^2 < \infty$ and

$$\mathsf{E}(W_T^2) = a^2 p_a + b^2 p_b = -ab \Rightarrow \mathsf{E}T = -ab$$

4 Question 4

We assume that $g(u,t)$ satisfies

$$g(u,h) = \int \pi(v, s \mid u) g(v, t+s) dv \tag{2}$$

$$\begin{aligned}
\mathsf{E}(Z_{t+h} \mid \mathcal{F}_t) &= \mathsf{E}(Z_{t+h} \mid X_t = u) \\
&= \mathsf{E}[g(X_{t+h}, t+h) \mid X_t = u) \\
&= \int g(v, t+h) \pi(v, h \mid u) dv = g(u,t)
\end{aligned}$$

First and last equalities are derived from the fact that X is a Markov process and (2) respectively.

For the case $g(x,t) = x^2 - t$,

$$\begin{aligned}
\int \pi(x, s \mid u) g(x, t+s) dx &= \int \sqrt{\frac{1}{2\pi s}} \exp\left\{-\frac{(x-u)^2}{2s}\right\} \left[x^2 - (t+s)\right] dx \\
&= \int \sqrt{\frac{1}{2\pi s}} \exp\left\{-\frac{(x-u)^2}{2s}\right\} x^2 dx - (t+s) \\
&= s + u^2 - (t+s) = u^2 - t = g(u,t)
\end{aligned}$$

Similarly for $g(x,t) = \exp\left\{\lambda x - \frac{\lambda^2 t}{2}\right\}$, use your wonderful integration skills to show that

$$\begin{aligned}
\int \pi(x, s \mid u) g(x, t+s) dx &= \int \frac{1}{\sqrt{2\pi s}} \exp\left\{-\frac{(x-u)^2}{2s}\right\} \exp\left\{\lambda x - \frac{\lambda^2}{2}(t+s)\right\} dx \\
&= g(u,t)
\end{aligned}$$

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Question 5

We can write

$$B = \sum_{j=1}^{n} b_j B_{t_j} \quad \text{for fixed } b_j$$
$$= \sum_{j=1}^{n} b_j (W_{t_j} - t_j W_1)$$
$$= \sum_{j=1}^{n} b_j W_{t_j} - W_1 \sum_{j=1}^{n} b_j t_j$$

Sum (and difference) of Gaussians is Gaussian. So B is Gaussian. This means that any linear combination (finite) of B_t is gaussian. So the collection $(B_{t_1}, \ldots, B_{t_n})$ is multivariate Guassian and B_t is a Gaussian process. The mean function is

$$\mathsf{E}(B_t) = \mathsf{E}(W_t) - t\mathsf{E}(W_1) = 0$$

and the autocovariance function is given by

$$\begin{aligned} \mathsf{Cov}(B_t, B_s) &= \mathsf{E}[(W_t - tW_1)(W_s - sW_1)] \\ &= \mathsf{E}(W_t W_s) + st\mathsf{E}(W_1^2) - t\mathsf{E}(W_1 W_s) - s\mathsf{E}(W_1 W_t) \\ &= s + st - ts - st = s(1-t) \end{aligned}$$

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Question 6

For simplicity, let

$$B_k = W_{k2^{-n}t} - W_{(k-1)2^{-n}t}$$

Since W_t is a standard Weiner process, $B_k \overset{\text{IID}}{\sim} \text{Gaussian}(0, 2^{-n}t)$. Then by the Law of Large Numbers,

$$\begin{aligned} \sum_{k=1}^{2^n} |B_k| &\to 2^n \mathsf{E}(|B_k|) \\ &= 2^{2^{-1}(n+1)} \left(\frac{t}{\pi}\right)^{1/2} \to \infty \text{ as } n \to \infty \end{aligned}$$

So it is proven that

$$\sum_{k=1}^{2^n} |B_k| \to \infty$$

Now let Δ_{nk} and U_{nk} be as defined in the assignment. Equation (7) in the assignment can be rewritten as

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} (U_{nk} + t2^{-n}) = t \quad \text{or equivalently} \quad \sum_{k=1}^{2^n} U_{nk} \to 0$$

The following are true:

$$\begin{aligned} \mathsf{E}U_{nk} &= \mathsf{E}(\Delta_{nk}^2 - t2^{-n}) = 0 \quad \text{recall } \Delta_{nk} \overset{\text{IID}}{\sim} \text{Gaussian}(0, 2^{-n}t) \\ \mathsf{E}(U_{nk}^2) &= \mathsf{E}(\Delta_{nk}^4 + 2t \cdot 2^{-n}\Delta_{nk}^2 + t^2 2^{-2n}) \\ &= 3t^2 2^{-2n} - 2t^2 2^{-2n} + t^2 2^{-2n} = \frac{2t^2}{2^{2n}} \to 0 \end{aligned}$$

as $n \to \infty$ and so L^2 convergence follows. The last equality follows from the hint.

By Chebyshev's inequality, we have that

$$\begin{aligned} \mathsf{P}\left(\left|\sum_{k=1}^{2^n} U_{nk}\right| > \epsilon\right) &\le \frac{\mathsf{E}\left[\left(\sum_{k=1}^{2^n} U_{nk}\right)^2\right]}{\epsilon^2} \\ &= \frac{2t^2}{\epsilon 2^n} \quad \text{due to iid property} \\ \Rightarrow \sum_n \mathsf{P}\left(\left|\sum_{k=1}^{2^n} U_{nk}\right| > \epsilon\right) &\le \frac{2t^2}{\epsilon^2} < \infty \end{aligned}$$

Finally, apply the Borel-Cantelli Lemma,

$$\mathsf{P}\left(\left|\sum_{k=1}^{2^n} U_{nk}\right| > \epsilon \text{ i.o.}\right) = 0$$

and so for large n,

$$\mathsf{P}\left(\left|\sum_{k=1}^{2^n} U_{nk}\right| < \epsilon\right) = 1 \Rightarrow \mathsf{P}\left(\lim_{n\to\infty}\left|\sum_{k=1}^{2^n} U_{nk}\right| < \epsilon\right) = 1$$

Then, we have that

$$\mathsf{P}\left(\lim_{n\to\infty} \sum_{i=1}^{2^n} U_{nk} = 0\right) = 1 \text{ implies } \sum_{k=1}^{2^n} U_{nk} \xrightarrow{a.s.} 0$$

7

Question 7

$$
\begin{aligned}
\mathsf{P}\left(\max_{0\le s\le T} W_s \ge c\right) &= 1 - \mathsf{P}\left(\max_{0\le s\le T} W_s < c\right) \\
&= 1 - \mathsf{P}\left(\max_{0\le s\le T} W_s \ge c\right) \\
&= 1 - \mathsf{P}\left(W_T \text{ failed to pass } c \text{ and } W_T < c\right) \\
&= 1 - \left[\mathsf{P}\left(W_T < c\right) - \mathsf{P}\left(W_T < c \text{ and } W_T \text{ passed } c\right)\right] \\
&= 1 - \left[\mathsf{P}\left(W_T < c\right) - \mathsf{P}\left(W_T > c \text{ and } W_0 = 0\right)\right] \\
&= 1 - \Phi\left(\frac{c}{\sqrt{T}}\right) + 1 - \Phi\left(\frac{c}{\sqrt{T}}\right) = 2\left[1 - \Phi\left(\frac{c}{\sqrt{T}}\right)\right]
\end{aligned}
$$

Third to last equality follows from reflection principle.

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Question 8

For first case, let W be the waiting time so that $0 \le W \le \Delta$. It can be easily shown using basic definition of expected value for continuous random variables that

$$\mathsf{E}W = \frac{\Delta}{2}$$

For the second case, the mean measure is $\frac{\text{Leb}}{\Delta}$ and for waiting time W

$$
\begin{aligned}
\mathsf{P}(W > t) &= \mathsf{P}(N(A) = 0 \text{ where } A = (0, t)) \\
&= \exp\left(-\frac{\text{Leb}(A)}{\Delta}\right) \\
&= \exp\left(-\frac{t}{\Delta}\right)
\end{aligned}
$$

This gives the kernel of Exponential(Δ^{-1}) and so $\mathsf{E}W = \Delta$.

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Question 9

Fix some arbitrary point and let D be the distance from the point to the closest tree. Further let A_r be the circle with radius r and centered at the fixed point. Then

$$\lambda \text{Leb}(A_r) = \pi r^2 \lambda \Rightarrow N(A_r) \sim \text{Poisson}(\pi r^2 \lambda)$$

Then we have that

$$\mathsf{P}(D > r) = \mathsf{P}(N(A_r) = 0) = \exp(-\pi r^2 \lambda)$$

One-minus it and we get the CDF of D. For the second part, it's typical to treat the line of sight as a rectangle with arbitrary length ℓ and width $2a$. If there is a tree in this rectangular area, call it A, the view to the east is blocked. So we have

$$\lambda \text{Leb}(A) = 2a\ell\lambda \Rightarrow N(A) \sim \text{Poisson}(2ar\lambda)$$

Thus,

$$\mathsf{P}(D > \ell) = \mathsf{P}(N(A) = 0) = \exp(-2a\ell\lambda) \Rightarrow D \sim \text{Exponential}(2a\lambda)$$

and so $\mathsf{E}D = (2a\lambda)^{-1}$

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Question 10

$$\begin{aligned}
&\mathsf{P}\left(N(A_1) = a_1 \cap \ldots \cap N(A_k) = a_k\right) \\
&= \mathsf{P}\left(N(A_1) = a_1\right) \prod_{i=2}^{k} \mathsf{P}\left(N(A_i) = a_i \,\middle|\, \bigcap_{j=1}^{i-1} \{N(A_j) = a_j\}\right) \\
&= \mathsf{P}\left(N(A_1) = a_1\right) \prod_{i=2}^{k} \mathsf{P}\left(N(A_i) - N(A_{i-1}) = a_i - a_{i-1}\right) \\
&= \mathsf{P}\left(N(A_1 + t) = a_1\right) \prod_{i=2}^{k} \mathsf{P}\left(N(A_i + t) - N(A_{i-1} + t) = a_i - a_{i-1}\right) \\
&= \mathsf{P}\left(N(A_1 + t) = a_1\right) \prod_{i=2}^{k} \mathsf{P}\left(N(A_i + t) = a_i \,\middle|\, \bigcap_{j=1}^{i-1} \{N(A_j + t) = a_j\}\right) \\
&= \mathsf{P}\left(N(A_1 + t) = a_1 \cap \ \ldots \ \cap N(A_k + t) = a_k\right)
\end{aligned}$$

Second equality and third equality follow from independent increments and stationary increments respectively.

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Question 11

Given regions A and B, this measure characterizes the process on overlapping parts of A and B. In the stationary case, it is 0. It is also 0 for Poisson process. To see this, we know that

$$\mathsf{Cov}(N(A), N(B)) = 2^{-1}[\mathsf{Var}(N(A)+N(B)) - \mathsf{Var}(N(A)) - \mathsf{Var}(N(B))]$$

We know $N(A)$ and $N(B)$ are Poisson with rates $\lambda\text{Leb}(A)$ and $\lambda\text{Leb}(B)$ respectively. Now use the MGF of $N(A)$ and $N(B)$ to derive the MGF of $N(A) + N(B)$. It turns out it is the MGF of a

$$\text{Poisson}(\lambda\text{Leb}(A) + \lambda\text{Leb}(B))$$

Thus,

$$\mathsf{Cov}(N(A), N(B)) = 2^{-1}[\mathsf{Var}(N(A)+N(B)) - \mathsf{Var}(N(A)) - \mathsf{Var}(N(B))] = 0$$

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Question 12

Suppose the Poisson random measure has mean measure $\lambda\text{Leb}(\cdot)$ and hence the avoidance measure is given by

$$V(A) = \mathsf{P}(N(A) = 0) = e^{-\lambda\text{Leb}(A)}$$

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Question 13

Define $A_t = \{\text{points } p : \|p\| \le t\}$. Denote the overall intensity λ. By the definition of homogeneous Poisson process, $N(A_t)$ is Poisson with rate λV_{A_t} where V_{A_t} is the volume of A_t. But

$$V_{A_t} = v_d t^d$$

Thus,

$$K(t) = \frac{\mathsf{E}[N(A_t)]}{\lambda} = \frac{\lambda \cdot v_d \cdot t^d}{\lambda} = v_d \cdot t^d$$