**Plan** Fun with White Noise Part II

0. Brief Review and Questions
1. More on the Random Walk
2. Recurrence and Transience
3. The Poisson Process
5. Brownian Motion

*Next Time: Martingales, ... Time will tell*

Reading: G&S 5.6–5.8
Homework 2 due next Tuesday

**Review Definition 1.** Stochastic Process I – Collection of Random Variables
Let $\mathcal{T}$ be an index set and $(\mathcal{X}, \mathcal{G})$ be a measurable space. A *stochastic process* is a collection of $\mathcal{X}$-valued random variables $(X_t)_{t \in \mathcal{T}}$.

**Review Definition 2.** Stochastic Process II – Random Functions
Let $\mathcal{T}$ be an index set and $(\mathcal{X}, \mathcal{G})$ be a measurable space. A *stochastic process* $X$ is random function defined on $\mathcal{T}$ and taking values in $\mathcal{X}$; that is, $X\colon \Omega \to \mathcal{X}^{\mathcal{T}}$. The realized functions $X(\omega)$ are called *sample paths* of the process.

**Notation 3.** We will write the values $X_t$ and $X(t)$ interchangeably as is most convenient. The realized values are $X_t(\omega)$ or $X(t, \omega)$ equivalently.

**Reminder: Rigor Alert 4.** In order to be precise, $X$ needs to be defined in some measurable space $(\mathcal{X}^{\mathcal{T}}, \mathcal{H})$ for some $\sigma$-field $\mathcal{H}$. With proper attention to detail, we can construct the $\sigma$-field, but we will discuss that only as needed in this course.

**Process 5.** Let $(X_n)_{n \in \mathbb{Z}_\oplus}$ be a sequence of independent, identically distributed, mean zero random variables. We will call this a *discrete-time white noise process.*

**Examples 6.** (a) Bernoulli process, (b) Radamacher process, (c) Gaussian white noise, (d) Exponential waiting times.

**Process 7.** Let $(X_n)$ be a white noise process and $S_0$ be an arbitrary random variable. Then, $(S_n)_{n \in \mathbb{Z}_\oplus}$, defined by

$$S_n = S_0 + \sum_{i=1}^{n} X_i, \tag{1}$$

is called a *random walk.*

**Examples 8.**

1. First return to zero (extended).
2. Hitting Time

**Theorem 9.** (The Lagrange Inversion Formula). Let $F(u)$ and $G(u)$ be formal power series in $u$, with $G(0) = 1$. Then there is a unique formal power series $U = U(z)$ that satisfies the functional equation

$$U = zG(U).$$

Moreover, the value $F(U(z))$ of $F$ at that "root" $U = U(z)$, when expanded in a power series in $z$ about $z = 0$, satisfies

$$[z^n]F(U(z)) = \frac{1}{n}[u^{n-1}](F'(u)G^n(u)).$$

**Example 10.**

1. Gambler's Ruin (several ways)

**Process 11.** Thus, derived from a random walk, we get a process $(N_t)_{t \geq 0}$ with the following properties:

1. $N_0 = 0$
2. For $0 \leq s < t$, $N_t - N_s$ has a Poisson$\langle \lambda(t - s) \rangle$ distribution.
3. For $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots$, the random variables $N_{t_i} - N_{s_i}$ are independent.

This is called a (homogeneous) *Poisson process* with rate $\lambda$.

**Process 12.** Thus, derived from a white noise process, we get a process $(W_t)_{t \geq 0}$ with the following properties:

1. $W_0 = 0$
2. For $0 \leq s < t$, $W_t - W_s$ has a Normal$\langle 0, t - s \rangle$ distribution/
3. For $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots$, the random variables $W_{t_i} - W_{s_i}$ are independent.

We call this process a *Weiner process* or equivalently, a *Brownian Motion.*

**Question 13.** From these properties, what is $\mathsf{E} W_t W_s$?