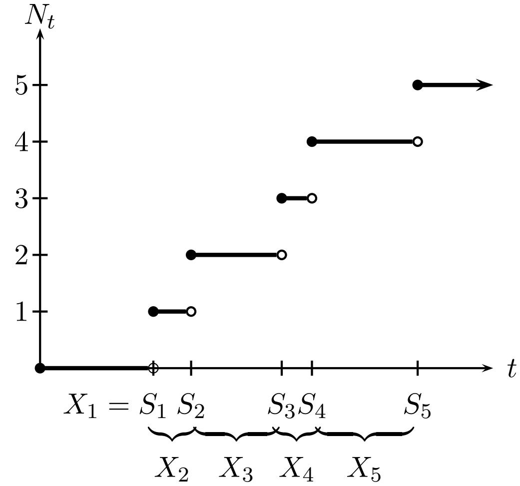
Plan Fun with White Noise, Last Part

1. The Poisson Process cont'd/revisited
3. Brownian Motion
3. Stationary Processes (if time allows – ha ha – or skip til later)

Next Time: Markov Processes (finally)

Reading: G&S 6.1, 6.2, generating function sheet
Homework 2 due Today
Homework 3 due next week

Figure 1.



Example 2. Poisson Process

Consider a random walk with $S_0 = 0$ and X_i IID Exponential$\langle\lambda\rangle$. (Note: If $W_i = X_i - \lambda$, $S_n = \sum_i W_i + n\lambda$, a sum of white noise plus drift.)

Then, S_n has a Gamma$\langle n, \lambda\rangle$ distribution.

Define $N_t = \max\{n \ge 0 \text{ such that } S_n \le t\}$.

A few properties directly from the definition:

1. $N_0 = 0$.
2. $N_t \ge n \iff S_n \le t$.
3. $N_t = n \iff S_n \le t$ and $S_{n+1} > t$.
4. If $s < t$, $N_t - N_s$ counts the number of "arrivals" between s and t. This random variable $N_t - N_s$ is often called the *increment* of the process over $(s, t]$.

We have the following.

$$\begin{aligned}
\mathsf{P}\{N_t = k\} &= \mathsf{P}\{S_k \le t, S_{k+1} > t\} && (1)\\
&= \mathsf{E}\,\mathsf{P}\{S_k \le t, S_{k+1} > t \mid S_k\} && (2)\\
&= \int_0^t \mathsf{P}\{X_{k+1} > t - u\}\,\mathsf{P}\{S_k \text{ near } u\} && (3)\\
&= \int_0^t e^{-\lambda(t-u)} \frac{\lambda^k u^{k-1}}{\Gamma(k)} e^{-\lambda u}\,du && (4)\\
&= e^{-\lambda t} \frac{\lambda^k}{k!}\,k \int_0^t u^{k-1}\,du && (5)\\
&= e^{-\lambda t} \frac{(\lambda t)^k}{k!}, && (6)
\end{aligned}$$

so N_t has a Poisson$\langle \lambda t \rangle$ distribution.

Next, we consider the distribution of $N_t - N_s$ for some $s < t$.

$$
\begin{aligned}
&\mathsf{P}\{N_t - N_s \geq k\} \\
&= \mathsf{P}\{S_{k+N_s} \leq t\} && (7) \\
&= \mathsf{P}\{S_{k+N_s} - S_{N_s} \leq t - S_{N_s}\} && (8) \\
&= \mathsf{P}\{S_{k+N_s} - S_{N_s} \leq t - s + (s - S_{N_s})\} && (9) \\
&= \mathsf{E}\,\mathsf{P}\{S_{k+N_s} - S_{N_s} \leq t - s + (s - S_{N_s}) \mid N_s, S_{N_s}\} && (10) \\
&= \sum_{n=0}^{\infty} \int_0^s \mathsf{P}\{S_{k+N_s} - S_{N_s} \leq t - s + (s - S_{N_s}) \mid N_s = n, S_{N_s} \text{ near } u\}\ \mathsf{P}\{N_s = n, S_{N_s} \text{ near } u\} && (11) \\
&= \sum_{n=0}^{\infty} \int_0^s \mathsf{P}\{S_{k+n} - S_n \leq t - s + (s - u) \mid N_s = n, S_{N_s} \text{ near } u\}\ \mathsf{P}\{N_s = n, S_{N_s} \text{ near } u\} && (12) \\
&= \sum_{n=0}^{\infty} \int_0^s \mathsf{P}\left\{X_{n+1} - (s-u) + \sum_{i=n+2}^{n+k} X_i \leq t - s \,\middle|\, N_s = n, S_{N_s} \text{ near } u\right\} \mathsf{P}\{N_s = n, S_{N_s} \text{ near } u\} && (13) \\
&= \sum_{n=0}^{\infty} \int_0^s \mathsf{P}\left\{X_{n+1} - (s-u) + \sum_{i=n+2}^{n+k} X_i \leq t - s \,\middle|\, X_{n+1} > s - u, S_n \text{ near } u\right\} \mathsf{P}\{N_s = n, S_{N_s} \text{ near } u\} && (14) \\
&= \sum_{n=0}^{\infty} \int_0^s \mathsf{P}\left\{X_{n+1} - (s-u) + \sum_{i=n+2}^{n+k} X_i \leq t - s \,\middle|\, X_{n+1} > s - u\right\} \mathsf{P}\{N_s = n, S_{N_s} \text{ near } u\} && (15) \\
&= \sum_{n=0}^{\infty} \int_0^s \mathsf{P}\left\{X_{n+1} + \sum_{i=n+2}^{n+k} X_i \leq t - s\right\} \mathsf{P}\{N_s = n, S_{N_s} \text{ near } u\} && (16) \\
&= \sum_{n=0}^{\infty} \int_0^s \mathsf{P}\{S_{n+k} - S_n \leq t - s\}\ \mathsf{P}\{N_s = n, S_{N_s} \text{ near } u\} && (17) \\
&= \sum_{n=0}^{\infty} \mathsf{P}\{S_{n+k} - S_n \leq t - s\}\ \mathsf{P}\{N_s = n\} && (18) \\
&= \sum_{n=0}^{\infty} \mathsf{P}\{S_k \leq t - s\}\ \mathsf{P}\{N_s = n\} && (19) \\
&= \mathsf{P}\{S_k \leq t - s\} && (20) \\
&= \mathsf{P}\{N_{t-s} \geq k\}\,. && (21)
\end{aligned}
$$

That step from (16) to (17) requires an argument. It follows from the relation

$$\mathsf{P}\{X > t + u \mid X > u\} = \mathsf{P}\{X > t\} \tag{22}$$

for a random variable X with an Exponential distribution. Thus we see that $N_t - N_s$ has the same distribution as N_{t-s}.

Now consider $s' < t' < s < t$. We could use the same basic argument to show that $N_t - N_s$ and $N_{t'} - N_{s'}$ are independent but it gets a bit messy. It's easier using generating functions.

Note that

$$G(z;\lambda) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} z^n = e^{\lambda(z-1)}. \tag{23}$$

From this, we get the G of N_t:

$$G_{N_t}(z) = e^{\lambda t(z-1)} \tag{24}$$

$$= e^{\lambda(t-s+s-t'+t'-s'+s'-0)(z-1)} \tag{25}$$

$$= e^{\lambda(t-s)(z-1)} e^{\lambda(s-t')(z-1)} e^{\lambda(t'-s')(z-1)} e^{\lambda(s'-0)(z-1)} \tag{26}$$

$$= G_{N_{t-s}}(z) G_{N_{s-t'}}(z) G_{N_{t'-s'}}(z) G_{N_{s'-0}}(z) \tag{27}$$

$$= G_{N_t - N_s}(z) G_{N_s - N_{t'}}(z) G_{N_{t'} - N_{s'}}(z) G_{N_{s'} - N_0}(z). \tag{28}$$

But $N_t = N_t - N_s + N_s - N_{t'} + N_{t'} - N_{s'} + N_{s'} - N_0$, and this equality of generating functions implies that the components are independent. (There's a brief argument needed, which I'll show you, to make this rigorous.) Conversely, if we show independence first, we could show equality in distribution with the same relation.

Thus, from a white noise random walk, we get the following process.

Process 3. Let $(N_t)_{t \geq 0}$ be a process with the following properties:

1. $N_0 = 0$
2. For $0 \leq s < t$, $N_t - N_s$ has a Poisson$\langle \lambda(t-s) \rangle$ distribution.
3. For $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_m < t_m$, the random variables $N_{t_i} - N_{s_i}$ are independent.

This is called a (homogeneous) *Poisson process* with rate λ.

Property 2 has two parts. The first is the specific distribution of the increment $N_t - N_s$. The second is that the distribution of the increment depends on time only through $t - s$. This is the property of *stationary increments*: the distributions of increments in two time intervals of the same length are equal.

Property 3 is called *independent increments*: the increments in disjoint time intervals are stochastically independent.

Poisson processes are examples of both *renewal processes* and *point processsses*, both of which we will study later.

Definition 4. Let $L^2(0,1)$ be (up to some formal details) the set of functions g on $(0,1)$ such that $\int_0^1 g^2 < \infty$.

A complete, orthonormal basis (ψ_n) for $L^2(0,1)$ is a countable subset of $L^2(0,1)$ such that $\int \psi_j \psi_k = \delta_{jk}$ and for any $g \in \mathcal{L}^2(0,1)$, we can find $c_n = \int g\psi_n$ such that

$$\int \left(g - \sum_{k=0}^{n} c_k \psi_k \right)^2 \to 0, \tag{29}$$

as $n \to \infty$.

Example 5. Brownian Motion

Define a process $(\xi_t)_{t \geq 0}$ as follows. Let $(A_n)_{n \geq 0}$ be a standard normal white noise Process, i.e., A_n are IID Normal$\langle 0, 1 \rangle$. Define

$$\xi_t(\omega) = \sum_{n=0}^{\infty} A_n(\omega)\psi_n(t), \tag{30}$$

for a particular complete orthonormal basis $(\psi_)$.

Taking some liberties (we'll see a more formal derivation another time), we will think of ξ_t as a continuous white noise process meaning that in some formal sense we should have $\mathsf{E}\xi_t = 0$ and $\mathsf{E}\xi_s\xi_t = \delta(s-t)$. Arguing loosely, this works:

$$\mathsf{E}\xi_t = \sum_n \mathsf{E}A_n \psi_n(t) = 0 \tag{31}$$

$$\mathsf{E}\xi_s\xi_t = \sum_{n,m} \mathsf{E}A_n A_m \psi_n(t)\psi_m(s) \tag{32}$$

$$= \sum_n \psi_n(t)\psi_n(s), \tag{33}$$

which "makes sense" when $s \neq t$ because

$$\int dt\, \mathsf{E}\xi_s\xi_t = \sum_n \psi_n(s) \int dt\, \psi_n(t) = 0. \tag{34}$$

But don't take that last calculation too seriously.

Now, just as we got a random walk process by taking cumulative sums of a discrete white noise process, we can see what we get when we take cumulative *integrals* of a *continuous* white noise process.

Define

$$W_t = \int_0^t \xi_s\, ds = \sum_{n=0}^{\infty} A_n \int_0^t \psi_n(s)\, ds, \tag{35}$$

where we choose a specific basis ψ_n.

Definition 6. Define $H = 1_{(0,1/2]} - 1_{(1/2,1]}$. Then let

$$H_{jk}(t) = 2^{j/2} H(2^j t - k). \tag{36}$$

Then, $H_0 = 1_{(0,1]}$ and H_{jk} for $j \geq 0$, $k = 0, \ldots, 2^j - 1$ form a complete orthonormal basis for $L^2(0,1)$. It is called the *Haar basis*.

To see this note that $\int H_0^2 = 1$, $\int H_{jk} H_{j'k'} = \delta_{jj'}\delta_{kk'}$, and $\int H_{jk} = 0$. And we have that $\alpha H_0 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \beta_{jk} H_{jk}$ gives a representation for all piecewise constant functions on dyadic intervals of length 2^{-J}. (We'll have a cool martingale proof of this another time.)

Example 5 cont'd Now order the Haar functions $(H_0, H_{00}, H_{10}, H_{11}, H_{20}, H_{21}, H_{22}, H_{23}, \ldots)$ and label these as ψ_n for $n \geq 0$. (For $2^j \leq n < 2^{j+1}$ and $j \in \mathbb{Z}_+$, take $k = n - 2^j$ and let $H_n \equiv H_{jk}$.)

For $n \geq 1$,

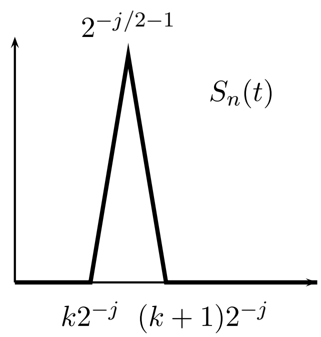
$$\int_0^t H_n(s)\, ds \equiv S_n(t), \tag{37}$$

called the Schauder function.

It follows that

$$W_t = \sum_{n \geq 0} A_n S_n(t). \tag{38}$$

Figure 7.



$2^j \leq n < 2^{j+1}$
$k = n - 2^j$

Lemma 8. Let (a_k) be a real sequence that satisfies $|a_k| = O(k^\gamma)$ for some $0 \leq \gamma < 1/2$. Define $f(t) = \sum_{k \geq 0} a_k S_k(t)$ and $f_n(t)$ be the corresponding partial sum. Then $f_n \to f$ uniformly on $(0, 1)$, meaning that $\sup_{0<t<1} |f_n(t) - f(t)| \to 0$.

Lemma 9. A standard normal white noise sequence A_n satisfies $|A_n| = O(\sqrt{\log n})$ with probability 1.

Lemma 10. If $0 \leq s, t \leq 1$,

$$\sum_{n \geq 0} S_n(s) S_n(t) = \min(s, t). \tag{39}$$

Proof of Lemma 10 Let $\phi_s = 1_{[0,s]}$. Then, if $s \leq t$,

$$s = \int_0^1 \phi_t \phi_s = \sum_{n \geq 0} a_n b_n \tag{40}$$

where

$$a_n = \int_0^1 \phi_t H_n = S_n(t) \tag{41}$$

$$b_n = \int_0^1 \phi_s H_n = S_n(s). \tag{42}$$

Example 5 cont'd So, W_t as defined exists as a random function. We get the following:

$$\mathsf{E}W_t = \sum_{n\geq 0} \mathsf{E}A_n S_n(t) = 0 \tag{43}$$

$$\mathsf{E}W_t^2 = \sum_{n\geq 0} \mathsf{E}A_n^2 S_n(t) = t \tag{44}$$

$$\mathsf{E}W_s W_t = \sum_{n,m\geq 0} \mathsf{E}A_n A_m S_n(t) S_m(s) = \min(s,t) \tag{45}$$

and for $u < s < t$,

$$\mathsf{E}(W_t - W_s)W_u = \sum_{n,m\geq 0} \mathsf{E}A_n A_m (S_n(t) - S_n(s)) S_m(u) = \min(t,u) - \min(s,u) = 0. \tag{46}$$

Moreover, using the characteristic generating functions with $s \leq t$,

$$\begin{aligned}
\mathsf{E}e^{i\lambda(W_t - W_s)} &= \mathsf{E}e^{i\lambda \sum_n A_n (S_n(t) - S_n(s))} &(47)\\
&= \prod_{n=0}^{\infty} \mathsf{E}e^{i\lambda A_n (S_n(t) - S_n(s))} &(48)\\
&= \prod_{n=0}^{\infty} e^{-\frac{\lambda^2}{2}(S_n(t) - S_n(s))^2} &(49)\\
&= e^{-\frac{\lambda^2}{2}\sum_n (S_n(t) - S_n(s))^2} &(50)\\
&= e^{-\frac{\lambda^2}{2}(t - 2s + s)} &(51)\\
&= e^{-\frac{\lambda^2}{2}(t - s)}, &(52)
\end{aligned}$$

using normality of the A_ns. Hence, $W_t - W_s$ has a Normal$\langle 0, t-s\rangle$ distribution. We get the following process.

Process 11. Thus, derived from a white noise process, we get a process $(W_t)_{t\geq 0}$ with the following properties:

1. $W_0 = 0$
2. For $0 \leq s < t$, $W_t - W_s$ has a Normal$\langle 0, t-s\rangle$ distribution.
3. For $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots$, the random variables $W_{t_i} - W_{s_i}$ are independent.

We call this process a *Weiner process* or equivalently, a *Brownian Motion.*

Definition 12. A stochastic process $X = (X_t)$ is called *strongly stationary* if for any $n \geq 1$, any $t_1, \ldots, t_n$, and any h, the two vectors

$$(X_{t_1}, \ldots, X_{t_n}) \text{ and } (X_{t_1+h}, \ldots, X_{t_n+h}) \tag{53}$$

have the same distribution.

Definition 13. A stochastic process $X = (X_t)$ is called *weakly stationary* (aka *second-order stationary*) if for any t_1, t_2 and h,

$$\mathsf{E}X_{t_1} = \mathsf{E}X_{t_2} \tag{54}$$

$$\mathsf{Cov}(X_{t_1}, X_{t_2}) = \mathsf{Cov}(X_{t_1+h}, X_{t_2+h}). \tag{55}$$

Examples 14.

1. Is the white noise process stationary? In which sense?
2. What about the Poisson process? Weiner Process?
3. ARMA processes

Definition 15. If X is a (weakly) stationary process, then three important functions that characterize the process:

1. The *autocovariance function* R is defined by

$$R(t) = \mathsf{Cov}(X_0, X_t). \tag{56}$$

2. The *autocorrelation function* ρ is defined by

$$\rho(t) = \mathsf{Cor}(X_0, X_t) = \frac{R(t)}{R(0)}. \tag{57}$$

3. The *spectral density* is defined when ρ has the representation

$$\rho(t) = \int e^{it\lambda} dF(\lambda) \tag{58}$$

   for some distribution function F. The spectral density is the function $f(\lambda) = F'(\lambda)$.

Example 16. White noise: what's in a name?