Plan Limit Theory (Countable-state case primarily)

1. Recurrence and Transience
2. Invariant Distributions
3. Limit Theorems

Next Time: General-state analogues, Extended Examplse

Reading: G&S 6.4, 6.5, 6.6
Homework 4 on-line tomorrow (sorry)
Homework solutions on-line up to date

Review Theorem 1. *The Chapman-Kolmogorov Equations.*
Let $n, m \geq 0$. For a transition probability matrix P, we have:

$$P^{n+m} = P^n \cdot P^m, \qquad \text{(that is, } P^{n+m}(s,s') = \sum_{u \in \mathcal{S}} P^n(s,u) P^m(u,s')\text{)}. \tag{1}$$

For general transition kernels:

$$P^{n+m}(s,A) = \int_{\mathcal{S}} P^n(s,du) P^m(u,A). \tag{2}$$

Review Theorem 2. If X is Markov chain on countable state space $\mathcal{S}$, then we can write

$$\mathcal{S} = D \cup \bigcup_i C_i, \tag{3}$$

where the sets are disjoint and each C_i is an absorbing, communicating class for the chain X.

Review Proposition 3. If $C \subset \mathcal{S}$ is an absorbing, communicating class for a Markov chain X, then there exists an irreducible Markov Chain X^C with state-space C and whose transition probability kernel is given by $P_C(x,A) \equiv P(x, A \cap C)$ for $x \in C$.

Review Definition 4. For any state $s \in \mathcal{S}$, define the *period* of s by

$$d(s) = \gcd\{n \geq 1 \text{ such that } P^n(s,s) > 0\}. \tag{4}$$

This implies that $P^n(s,s) = 0$ unless $n = md(s)$ for some $m \in \mathbb{Z}_+$. An irreducible Markov Chain is said to be *aperiodic* if $d \equiv 1$.

Review Theorem 5. Let X be an irreducible, countable-state Markov chain with common period d. Then, there are disjoint sets $U_1, \ldots, U_d \subset \mathcal{S}$ such that

$$\mathcal{S} = \bigcup_{k=1}^{d} U_k, \tag{5}$$

and

$$\mathsf{P}(x, U_{k+1}) = 1 \quad \text{for } x \in U_k, \qquad k = 0, \ldots, d-1 \pmod{d}. \tag{6}$$

The sets $U_1, \ldots, U_d$ are called *cyclic classes* of X because X cycles through them successively.

Useful Random Variables 6. Let X be a general-state Markov chain with state space $\mathcal{S}$. Let $A \subset \mathcal{S}$. Define

$$T_A = \inf\{n \geq 1 \text{ such that } X_n \in A\} \tag{7}$$

$$S_A = \inf\{n \geq 0 \text{ such that } X_n \in A\} \tag{8}$$

$$O_A = \sum_{n=1}^{\infty} 1\{X_n \in A\}. \tag{9}$$

These are called, respectively, the *first return time*, the *first hitting time*, and the *occupation time* of A. If X_n never returns to or hits A, we take $T_A = \infty$ and $S_A = \infty$ respectively. As we will see, these random variables provide a great deal of information about the behavior of the chain.

For $A, B \subset S$ and $s \in \mathcal{S}$, define

$$R(s, A) = \mathsf{P}_s\{T_A < \infty\} \tag{10}$$

$$M(s, A) = \mathsf{E}_s T_A \tag{11}$$

$$H(s, A) = \mathsf{P}_s\{S_A < \infty\} \tag{12}$$

$$O(s, A) = \mathsf{E}_s O_A \tag{13}$$

$$P^n_{!A}(s, B) = \mathsf{P}_s\{X_n \in B, T_A \geq n\}. \tag{14}$$

These are the return time probabilities, hitting probabilies, expected occupation times for the set A and taboo probabilities for the set B avoiding A, when the chain starts in state s.

Note that

$$O(s, A) = \sum_{n=1}^{\infty} P^n(s, A) \tag{15}$$

$$R(s, A) = \sum_{n=1}^{\infty} P^n_{!A}(s, A). \tag{16}$$

Definition 7. The random variables T_A and S_A in the last item are stopping times, meaning that $\{T = n\} \in \sigma(X_0, \ldots, X_n)$ for every n.

Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ be the history of the chain up to time n. For a stopping time T, we can define the information in the chain up to time T – the history up to time T – as a σ-field $\mathcal{F}_T$ defined as follows:

$$\mathcal{F}_T = \{A \in \mathcal{F} \text{ such that } A \cap \{T = n\} \in \mathcal{F}_n, \text{ for all } n \in \mathbb{Z}_{\oplus}\}. \tag{17}$$

Theorem 8. *The Strong Markov Property*

For any (discrete-time) Markov chain X and a bounded, measurable function h on sample paths,

$$\mathsf{E}\left(h(X_{T+1}, X_{T+2}, \ldots) \mid \mathcal{F}_T\right) 1\{T < \infty\} = \mathsf{E}_{X_T}(h(X_1, X_2, \ldots) 1\{T < \infty\}, \tag{18}$$

where E_{X_T} corresponds to a chain whose initial distribution on $\mathcal{S}$ is the distribution of X_T. This is called the *Strong Markov Property*.

Definition 9. A set $A \subset \mathcal{S}$ is called *uniformly transient* if there exists $M < \infty$ such that $O(s, A) \leq M$ for all $s \in A$. A is called *transient* if $O(s, A) < \infty$ for all $s \in A$. A is called *recurrent* if $O(s, A) = \infty$ for all $s \in A$.

In particular, for a state $s \in \mathcal{S}$ of a *countable-state Markov chain*, we say that s is uniformly transient/transient/recurrent if $\{s\}$ is.

Theorem 10. For an irreducible, countable-state Markov chain X, either $O(s, s') < \infty$ for all $s, s' \in \mathcal{S}$, in which case we say that X is recurrent, or $O(s, s') = \infty$ for all $s, s' \in \mathcal{S}$ in which case we say that X is transient.

Proof. Then since for any u, v, $u \to s$ and $s' \to v$, we can find an ℓ, m such that $P^\ell(u, s) > 0$ and $P^m(s', v) > 0$. Hence,

$$\sum_n P^{\ell+m+n}(u, v) > P^\ell(u, s) \left[\sum_n P^n(s, s') \right] P^m(s', v). \tag{19}$$

It follows that $O(s, s') = \sum_{n=1}^{\infty} P^n(s, s') = \infty$ implies $O(u, v) = \infty$ and $O(u, v) < \infty$ implies $O(s, s') < \infty$. But both pairs of states were arbitrary, so the theorem is proved.

Theorem 11. Suppose X is a countable-state Markov chain on $\mathcal{S}$. For any $s \in \mathcal{S}$, $O(s, s) \equiv O(s, \{s\}) = \infty$ if and only if $R(s, s) \equiv R(s, \{s\}) = 1$.

Hence, if X is irreducible, either $R(s, s') = 1$ for all $s, s' \in \mathcal{S}$ or $R(s, s) < 1$ for all $s \in \mathcal{S}$.

To prove this theorem, we'll use the same trick we used earlier in considering the return times of random walks. Notice that for any $s \in \mathcal{S}$ and any $n \geq 1$,

$$P^n(s, s) = \sum_{k=1}^{n} \mathsf{P}_s\left\{T_{\{s\}} = k\right\} P^{n-k}(s, s) = \sum_{k=0}^{n} \mathsf{P}_s\left\{T_{\{s\}} = k\right\} P^{n-k}(s, s), \tag{20}$$

where the latter follows because $\mathsf{P}_s\left\{T_{\{s\}} = 0\right\} = 0$.

Let $G_s(z) = \sum_n P^n(s, s) z^n$ and $R_s(z) = \sum_n \mathsf{P}_s\left\{T_{\{s\}} = n\right\} z^n$. Then, we get

$$G_s(z) = 1 + G_s(z) R_s(z) \quad \Longrightarrow \quad G_s(z) = \frac{1}{1 - R_s(z)}. \tag{21}$$

Because $R_s(1) = \mathsf{P}_s\left\{T_{\{s\}} < \infty\right\}$ (or letting $z \to 1$ to be careful about convergence), we get that

$$R(s, s) = 1 \iff O(s, s) = \infty. \tag{22}$$

By Theorem 10 and equation (22), we have either $R(s, s) < 1$ for all s or $R(s, s) = 1$ for all s. If the latter is true and $R(s, s') < 1$, then by irreducibility, we have $O(s', s) > 0$ and thus, for some n, $P^n_{!s'}(s', s) > 0$. This implies $R(s', s') < 1$, and the result follows by contradiction.

Examples 12.

1. Random Walk
2. Bounded Random Walk
3. Binomial Runs
4. Renewal Process and Forward Recurrence Time Chain

Definition 13. For $s \in \mathcal{S}$, recall $M(s, A) = \mathsf{E}_s T_A$. For $A = \{s\}$, $M(s, A) \equiv M(s, s)$. These are the expected return times to the set A and the state s.

Definition 14. Let X be a countable-state Markov chain on $\mathcal{S}$. If $s \in \mathcal{S}$ is a recurrent state, we call it *positive recurrent* if $M(s, s) < \infty$ and *null recurrent* if $M(s, s) = \infty$.

Definition 15. Let X be a countable-state Markov chain on $\mathcal{S}$ with transition probabilities P. A (σ-finite) measure π is an invariant measure for the chain if $\pi(s) \geq 0$ and

$$\pi(s') = \sum_s \pi(s) P(s, s'), \tag{23}$$

or in matrix terms

$$\boldsymbol{\pi} = \boldsymbol{\pi} \cdot P, \tag{24}$$

where we think of $\boldsymbol{\pi}$ as a "row vector".

An invariant measure π is an *invariant* or *stationary distribution*, if in addition it is a probability mass function on $\mathcal{S}$.

Motivation 16. The names "invariant" and "stationary" come from the above properties. The first stems from the fact that π does not change – is invariant – under the transition mechanism of the chain. The second comes fro the fact that if the chain is started with initial distribution π, then the distribution of X_n is given by $\pi \cdot P^n = \pi$. That is, the distribution of X_n does not change with n.

Definition 17. If X is an irreducible, recurrent, countable-state Markov chain on $\mathcal{S}$ and there exists a stationary distribution on $\mathcal{S}$, then X is said to be a positive recurrent. Otherwise, X is said to be null recurrent. Then, every state is, respectively, positive or null recurrent.

Theorem 18. If X is an irreducible, recurrent, countable-state Markov chain, then there exists an invariant *measure* ρ that is positive and unique up to constant multiples. The chain is positive recurrent if $\sum_s \rho(s) < \infty$ and null recurrent if $\sum_s \rho(s) = \infty$.

Theorem 19. If an irreducible, countable-state Markov chain has a stationary distribution π, then it is unique and $\pi_s = 1/M(s, s)$.

Remarks 20. 1. Note that if X admits a stationary distribution but is transient then $P^n(s, s') \to 0$ as $n \to \infty$, so

$$\begin{aligned} \pi_{s'} &= \sum_s \pi_s P(s, s') \leq \sum_{s \in S_0} \pi_s P(s, s') + \sum_{s \notin S_0} \pi_s & (25) \\ &\to \sum_{s \notin S_0} \pi_s \to 0, & (26) \end{aligned}$$

for any finite $S_0 \subset \mathcal{S}$ with the last limit being as $S_0 \uparrow S$. This is a contradiction, so X must be recurrent.

2. See the proofs in G&S section 6.4.

To translate, define $q(s, s') = \sum_n P^n_{!s}(s, s')$ for the taboo probabilies $P^n_{!s}(s, s')$. Notice that $M(s, s) = \sum_{s'} q(s, s')$. G&S use $\rho_{s'}(s) \equiv q(s, s')$ and $f_{s,s'}(n) \equiv \mathsf{P}_s\{T_{s'} = n\}$.

Examples 21.

- Doubly Stochastic, Finite Random Walk
- Canonical Random Walk
- Embedded MC for G/M/1 Queue